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EXERCISES

IN

CONCRETE GEOMETRY

WITH

SUPPLEMENTARY DISCUSSIONS

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PREFACE

KNOWLEDGE is useful only as it is used. It is not enough that the pupil learn the abstract principles of mathematics: he should make them real and practical by concrete application. He may be able to state a proposition correctly, to give all the whys and wherefores of its demonstration in the best logical form, and yet be utterly helpless if required to apply the principle thus obtained to the solution of a simple problem in arithmetic.

Many high school curriculums provide, during the closing year of the course, for a term's study of concrete geometry, which shall involve a general review of arithmetic and algebra, thus unifying and correlating all the mathematical studies pursued by pupils above the grammar grades. The plan is ideal, and is in line with the practical trend of modern educational thought. To meet the demands of such a course, it is hoped this work will be found peculiarly adapted.

The Manual is also offered as a supplementary text-book in geometry. Following the usual and logical order of the school geometries, whenever, in the regular course of study and instruction, a principle susceptible of concrete application is reached, exercises are at hand for the use of the learner. "Catches," puzzles, and impractical problems have been studiously avoided. The author believes that the exercises will be found sufficient in number, variety, and difficulty to meet the requirements of the class-room.

The book is made up largely from the author's classroom notes, and is the outcome of an experience of more than twenty-five years in teaching geometry in a state normal school, in which these and like exercises have been in constant use in his classes.

Chapters II and III contain such general hints and suggestions as the writer has found useful in leading pupils in the paths of original demonstration and construction. Special attention is asked to the method of employing tests in Chapter II, to the treatment of "Loci" and "Intersection of Loci" in Chapter III, and of "Limits" in Chapter IV.

The last chapter appeared originally, nearly as presented herein, as an article in Midland Schools. The pupil in solid geometry, however deficient in artistic skill, who carefully follows the directions given, will produce, at worst, diagrams sufficient for all ordinary purposes of demonstration.

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TO THE TEACHER

As a rule, results involving surds should be placed in the simplest radical form. Pupils stand in much greater need of drill in handling and simplifying radical quantities than of exercise in the arithmetical processes of extracting the square root and cube root of numbers. In problems in which the ratio of the circumference to the diameter of a circle is involved, let the answers be found in terms of π, without employing its numerical value.

The answers at the end of the book conform to the above suggestions.

EXERCISES IN CONCRETE GEOMETRY

CHAPTER I

BOOK I. FIGURES FORMED WITH STRAIGHT LINES.

1. Algebraic Review.

Surds.

1. Simplify the following Surds: $\sqrt{8}$, $\sqrt{12}$, $\sqrt{18}$, $\sqrt{20}$, $\sqrt{24}$, $\sqrt{27}$, $\sqrt{28}$, $\sqrt{32}$, $\sqrt{40}$, $\sqrt{44}$, $\sqrt{45}$, $\sqrt{48}$, $\sqrt{50}$, $\sqrt{52}$, $\sqrt{54}$, $\sqrt{56}$, $\sqrt{60}$, $\sqrt{63}$, $\sqrt{68}$, $\sqrt{72}$, $\sqrt{75}$, $\sqrt{76}$, $\sqrt{80}$, $\sqrt{84}$, $\sqrt{88}$, $\sqrt{90}$, $\sqrt{92}$, $\sqrt{96}$, $\sqrt{98}$, $\sqrt{99}$.

The above simplifications should be given orally and rapidly by the pupil.

2. Simplify $\sqrt{112}$, $\sqrt{120}$, $\sqrt{200}$, $\sqrt{288}$, $\sqrt{360}$, $\sqrt{404}$, $\sqrt{444}$, $\sqrt{500}$, $\sqrt{600}$, $\sqrt{1000}$, $\sqrt{1124}$, $\sqrt{1134}$, $\sqrt{1144}$, $\sqrt{2888}$, $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}$, $\sqrt{27}+\sqrt{75}-2\sqrt{3}$.

3. Simplify $\sqrt{\tfrac{1}{2}}$, $\sqrt{\tfrac{1}{3}}$, $\sqrt{\tfrac{2}{3}}$, $\sqrt{\tfrac{3}{4}}$, $\sqrt{\tfrac{1}{5}}$, $\sqrt{\tfrac{2}{5}}$, $\sqrt{\tfrac{3}{5}}$, $\sqrt{\tfrac{4}{5}}$, $\sqrt{\tfrac{1}{6}}$, $\sqrt{\tfrac{5}{6}}$, $\sqrt{\tfrac{1}{7}}$, $\sqrt{\tfrac{3}{7}}$, $\sqrt{\tfrac{5}{7}}$, $\sqrt{\tfrac{11}{7}}$, $\sqrt{\tfrac{2}{17}}$, $\sqrt{\tfrac{3}{20}}$, $\sqrt{\tfrac{1}{27}}$, $\sqrt{\tfrac{5}{32}}$, $\sqrt{\tfrac{11}{40}}$, $\sqrt{\dfrac{a}{b}}$, $\dfrac{\sqrt{ab}}{\sqrt{cd}}$, $\sqrt{\dfrac{a^2}{b^3}}$ $\sqrt{\dfrac{a^3}{b^5}}$, $\sqrt{\dfrac{a^3b^5}{c^7}}$, $\sqrt{\dfrac{a^2b^3c^4}{d^5e^6f^7}}$.

Quadratic Equations.

Solve the following:

1. $x^2+2x=8.$
2. $x^2+7x=18.$
3. $2x^2+5x=33.$
4. $3x^2+11x=14.$
5. $x^2+2x=7.$
6. $x^2+5x=11.$
7. $2x^2+9x=13.$
8. $3x^2-4x=1.$
9. $x^2+5ax=6a^2.$
10. $\frac{x}{a}+\frac{a}{x}=2.$
11. $x+\sqrt{x}=30.$
12. $x+1+\sqrt{x+1}=6.$
13. $\frac{x}{6}+\frac{6}{x}=9\frac{1}{9}.$
14. $\frac{x}{x-b}-\frac{x-b}{x}=1.$
15. $x^2+11+6\sqrt{x^2+11}=72.$

Solve the following simultaneous equations:

1. $\begin{cases} xy=12. \\ x+y=7. \end{cases}$
2. $\begin{cases} xy=6. \\ x-y=5. \end{cases}$
3. $\begin{cases} x+y=9. \\ x^2+y^2=41. \end{cases}$
4. $\begin{cases} x-y=3. \\ x^2+y^2=17. \end{cases}$
5. $\begin{cases} x^2+xy+y^2=7. \\ x+y=3. \end{cases}$
6. $\begin{cases} xy=a. \\ x+y=b. \end{cases}$
7. $\begin{cases} xy=a. \\ x-y=b. \end{cases}$
8. $\begin{cases} x+y=m. \\ x^2+y^2=m. \end{cases}$
9. $\begin{cases} xy=2p^2. \\ x^2+y^2=5p^2. \end{cases}$
10. $\begin{cases} x+y=a. \\ x^2-y^2=b. \end{cases}$

2. Preliminary.

Draw a straight line one foot long. After drawing the line free hand, test its accuracy by means of a foot measure.

Represent three straight lines by a, b, and c, making $a>b>c$.

Draw —

1. $2a$,
2. $a+b$,
3. $a+b-c$,
4. $a-b$,
5. $2a+3b$,
6. $a-b+c$,
7. $2a-2b$,
8. $\frac{1}{2}a$,
9. $a-\frac{1}{2}c$,
10. $\frac{1}{2}a+\frac{1}{4}b$.

3. Definitions.

The *Complement of an Angle* is the difference between 90° and the angle. The *Supplement of an Angle* is the difference between 180° and the angle.

1. What is the complement of 12°? of 59°? of 37°? of 32° 33′ 34″? of $a°$? of $(a+b)°$?

2. What is the supplement of 21°? of 56° 56′? of 139° 47′ 57″?

3. The complement of an angle is 80°, what is its supplement? The supplement of an angle is 153°, what is its complement?

4. How many degrees in an angle which is twice its complement? in an angle which is three times its complement?

5. The difference of two complementary angles is 8°, how many degrees in each angle?

6. The difference of two supplementary angles is 26°, how many degrees in each angle?

7. The difference of two supplementary angles is $a°$, how many degrees in each angle?

4. Theorem.

If one straight line meet another, the sum of the angles formed on one side of the line is equal to two right angles.

COR. 1. *The sum of all the angles formed by drawing lines to a point on one side of a straight line is equal to two right angles.*

COR. 2. *The sum of all the angles formed by drawing lines to a point is equal to four right angles.*

1. Three angles are formed by drawing straight lines from a point on one side of a straight line; the first contains 34°, the second 48°, how many degrees in the third?

2. A straight line CD is drawn to meet a straight line AB at D, so that the difference of the angles ADC and CDB equals 12°; how many degrees in each angle?

Four straight lines are drawn from a point forming the angles A, B, C, and D:

3. How many degrees in each angle if they are in the proportion of 1, 2, 3, and 4?

4. If they are in the proportion of 2, 3, 4, and 5?

5. If they are in the proportion of 3, 3, 3, and 6?

6. If they are in the proportion of 20, 21, 31, and 10?

7. If they are in the proportion of 1, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$?

8. If b is twice a, c is twice b, and d is twice c?

9. If b is $\frac{1}{2}$ a, c is 20° more than b, and d is equal to the sum of a, b, and c?

10. If b is 50° more than a, c is 10° less than a, and d is equal to the sum of a and c?

11. If b is 4 times a, c is five times a, and d is the complement of a?

12. If b is the complement of a, c is 50° more than a, and d is three times a?

13. Prove that the bisectors of a pair of supplementary adjacent angles meet at an angle of 90°.

14. Prove that the bisectors of a pair of opposite angles form a straight line.

5. Theorem.

The sum of the three angles of a triangle is equal to two right angles.

1. How many degrees in each angle of the triangle *ABC*, if they are in the proportion of 1, 3, and 8?

2. If they are in the proportion of 5, 6, and 7?

3. Of 17, 18, and 19?

4. Of 23, 31, and 36?

5. Of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$?

6. The sum of the two largest angles of a triangle is 150°, and the difference of the two smallest angles is 40°; how many degrees in each angle?

7. How many degrees in each angle of the triangle *ABC*, if angle *B* is twice angle *A*, and angle *C* is 20° more than angle *B*?

8. If angle *B* is 5 times angle *A*, and angle *C* is the complement of angle *A*?

9. If angle *B* is 7 times angle *A*, and angle *C* is the square of angle *A*?

10. If angle *B* is 12° less than angle *A*, and angle *C* is the square of angle *B*?

11. The side *BC* of the triangle *ABC* is produced to *D*. If the angle *A* is 44°, and the angle *B* is 57°, how many degrees in the angle *ACD*?

12. The angle *A* of the triangle *ABC* is 64°. The angles *B* and *C* are bisected by the lines *AO* and *CO*, how many degrees in the angle *BOC*?

13. In the above problem if the angle *A* is 100°, how many degrees in the angle *BOC*?

14. One of the acute angles of a right triangle is 34: a perpendicular is dropped from the vertex of the right

angle to the hypothenuse, how many degrees in each of the angles into which the right angle is divided?

15. The acute angles of a right triangle are bisected; how many degrees in the angle formed by the meeting of the bisectors?

16. The base *BC* of the triangle *ABC* is produced to *D*; the angle *A* is 44°, and the angle *B* is 54°; the bisectors of the angle *B*, and the angle *ACD* meet at *E*; how many degrees in the angle *E*?

17. How many degrees in the angle *E* (constructed as in 16), if angle *A* is 68° and angle *B* is unknown?

18. How many degrees in the angle formed by the bisector of an acute angle of a right triangle and the bisector of the exterior angle at the other extremity of the hypothenuse?

19. Prove that the angle formed by the bisector of an angle of a triangle, and the bisector of the exterior angle at the other extremity of the base, is always equal to one-half the vertical angle of the triangle.

20. Prove that if two parallel straight lines are cut by a transversal, the bisectors of the interior angles on the same side of the transversal meet at an angle of ninety degrees.

6. Theorem.

In an isosceles triangle, the angles opposite the equal sides are equal.

1. The vertical angle of an isosceles triangle is 12°, how many degrees in each of the angles at the base?

2. How many degrees in each of the angles at the base of an isosceles right triangle?

3. How many degrees in each of the angles of an equilateral triangle?

4. The vertical angle of an isosceles triangle is $2m$ degrees; how many degrees in each of the angles at the base?

5. The vertical angle A of the isosceles triangle ABC is 62°; a perpendicular CD to the side AB is drawn from the angle C, how many degrees in the angle DCB?

6. The vertical angle of the isosceles triangle ABC is 88°, the angle C is bisected by the line CD; how many degrees in the angle BDC?

7. How many degrees in the angle formed by the base of an isosceles triangle and the bisector of its vertical angle?

8. How many degrees in the angle formed by the bisectors of the base angles of an isosceles triangle, if its vertical angle is 8°? 18°? 100°? 170°?

9. In the triangle ABC, angles B and C are equal, and angle A is 46°; if CD is drawn perpendicular to AB, determine the number of degrees in the angle BCD; in the angle ACD.

10. If the vertical angle of an isosceles triangle is 52, determine the number of degrees in the angle formed by the bisector of one of the base angles and the side opposite that angle.

11. The bisectors of the angles of the isosceles triangle, one of whose base angles is 30°, meet at 0; find the number of degrees in each of the three angles formed about 0.

12. In the isosceles triangle ABC, the vertical angle A is 60° more than the square of the angle B. Find the number of degrees in angles of the triangle.

13. In the isosceles triangle ABC, the vertical angle A is 30° less than the angle B; how many degrees in each angle of the triangle?

14. How many degrees in the angle formed by the meeting of the bisectors of two angles of an equilateral triangle?

7. Theorems.

The sum of the interior angles of a polygon of n *sides is* $(2n - 4)$ *right angles.*

The sum of the exterior angles of a polygon is four right angles.

1. How many right angles in the sum of the interior angles of a polygon of 11 sides? of 17 sides? of 27 sides? of 47 sides? of 53 sides? of 65 sides? of 100 sides?

2. The sum of the interior angles of a polygon is 56 right angles; how many sides has the polygon?

3. How many sides has a polygon if the sum of its interior angles is six times the sum of its exterior angles?

4. In the trapezium $ABCD$, the angle A is 84° and the angle B 106°, how many degrees in the angle formed by the bisectors of angles C and D?

5. In the pentagon $ABCDE$, the angle A is 100°, the angle B, 120°, and the angle C, 96°; how many degrees in the angle formed by the bisectors of the angles D and E?

6. All the angles of a heptagon are equal; how many degrees in each angle?

7. The diagonals of an oblong intersect at an angle of 24°; find the number of degrees in each of the eight angles that the diagonals make with the sides?

8. Construct a square $ABCD$; lay off on the diagonal AD, a distance AE equal to AB. At E erect a perpendicular to AD, meeting BD at F; draw BE.

How many degrees in each of the following angles: *EDF*, *EFD*, *ABE*, *AEB*, *EBF*, *BEF*, *EFB*?

9. Prove that sum of the angles formed by producing all the sides of a pentagon till they meet, is equal to two right angles.

10. The interior angles of a pentagon are in the proportion of 1, 3, 3, 4, and 4; how many degrees in each angle?

11. The exterior angles of a quadrilateral are in the proportion of 5, 6, 7, and 12; how many degrees in each of the interior angles of the quadrilateral?

12. From a point 0 within an isosceles triangle whose vertical angle is 28°, a perpendicular is dropped to each of the sides; how many degrees in each of the angles about 0?

13. One angle of an isosceles trapezoid is 8°; find the number of degrees in each of the other angles of the trapezoid.

8. Miscellaneous.

1. Given n points, no three of which are in the same straight line; to find how many straight lines can be drawn to connect each point with one other point.

2. How many diagonals can be drawn in a polygon of 4 sides? of 5 sides? of 6 sides? of n sides? Derive formula.

3. How would you find the distance between two objects placed on opposite sides of the school building by taking measurements on the play ground.

Suggestion. — Employ the principle that the diagonals of a parallelogram bisect each other.

BOOK II. CIRCLES.

9. Theorems.

An inscribed angle is measured by one-half of the intercepted arc.

An angle formed by a chord and a tangent is measured by one-half of the intercepted arc.

An angle formed by two chords which intersect within the circumference is measured by one-half the intercepted arc plus one-half the arc intercepted by the sides produced.

An angle formed outside a circle by the meeting of two secants, of a secant and a tangent, or of two tangents, is measured by one-half the difference of the intercepted arcs.

1. An inscribed angle contains 28°, how many degrees in the intercepted arc?

2. An inscribed angle intercepts an arc of 42°, how many degrees in the angle?

3. Two angles of an inscribed triangle are 29° and 45°, how many degrees in each of the three arcs into which the circumference is divided?

4. Two angles of a circumscribed triangle are 72° and 82°, how many degrees in each of the three arcs into which the circumference is divided?

5. Two chords intersect in a circle; an opposite pair of arcs intercepted being 40° and 6°; find the number of degrees in the angle formed by the chords.

6. Two secants meeting without a circle cut the circumference so that the larger arc is 40°, and the smaller 6°; find the number of degrees in the angle.

7. Two secants meet at an angle of 35°; the large arc intercepted is 82°; find the smaller intercepted arc.

8. Two secants meet at an angle of 30°; the smaller arc intercepted is 5°, find the larger intercepted arc.

9. Two angles of an inscribed triangle are 62° and 66°; a triangle is circumscribed about the circle by drawing tangents through the vertices of the triangle; how many degrees in each angle of the circumscribed triangle?

10. Two angles of a circumscribed triangle are 68° and 70°; a triangle is inscribed in the circle by joining the points of tangency; how many degrees in each angle of the inscribed triangle?

11. If the given angles in Problem 9 are 65 and 75, find the number of degrees in each angle of the circumscribed triangle.

12. If the given angles in Problem 10 are 59 and 67, find the number of degrees in each angle of the inscribed triangle.

13. Prove that the sums of the opposite sides of a circumscribed quadrilateral are equal.

14. The sides AB, BC, and CD of a circumscribed quadrilateral are respectively, 20, 27, and 29; what is the length of the side AD?

15. The sides of a triangle circumscribed about a circle are 27, 28, and 29. Find the lengths of the segments intercepted between the vertices of the triangle and the points of tangency.

16. Find the lengths of the segments as in Problem 29, if the sides of the triangle are 40, 45, and 50; 70, 72, and 76.

17. Prove that the sums of the opposite angles of an inscribed quadrilateral are equal.

18. Two adjacent angles of an inscribed quadrilateral are 39° and 99°; find the number of degrees in each of the remaining angles.

10. Problems.

1. Construct an angle of 60°; of 30°; of 45°; of 120°; of 75°; of 150°; of 135°; of $22\frac{1}{2}$°.

2. Construct a triangle whose sides are in the proportion of 3, 4, and 5; in the proportion of 5, 6, and 7.

3. Construct a triangle having two of its sides in the proportion of 3 and 8, and the angle they include 15°.

4. Given the longest side of a triangle, and the angles in the proportion of 1, 2, and 3; to construct the triangle.

5. Given the shortest side of a triangle, and the angles in the proportion of 3, 4, and 5; to construct the triangle.

6. To construct upon a given hypothenuse an isosceles right triangle.

7. Given one of the equal sides of an isosceles right triangle, to construct the triangle.

8. To construct an isosceles triangle, having given its altitude twice its base.

9. To divide a given line into three equal parts; into five equal parts.

10. To find $\frac{3}{7}$ of a given line.

11. To divide a given line in the proportion of 2, 3, and 6; in the proportion of 1, 1, and 5; in the proportion of 1, $1\frac{1}{2}$, and $1\frac{1}{3}$.

12. Given the perimeter of a triangle, and the sides in the proportion of 2, 6, and 7; to construct the triangle.

13. To trisect a given semi-circumference.

14. To trisect a given quadrant.

15. Given the diagonals of a parallelogram in the proportion of 3 to 5, and the angle at which they intersect 30°; to construct the parallelogram.

16. Given one angle of a parallelogram 75°, and the sides in the proportion of 5 to 8, to construct the parallelogram.

17. Given the sides of a quadrilateral in the proportion of 1, 2, 3, and 4, and the angle formed by the two shortest sides $22\frac{1}{2}°$, to construct the quadrilateral.

18. Given the shorter diagonal of a rhombus, and the adjacent angles of the rhombus in the proportion of 1 to 2, to construct the rhombus.

19. Divide a given triangle into four equal triangles.

20. Divide a given parallelogram into eighteen equal triangles.

21. Given the diagonal of a square, to construct the square.

22. Given the altitude of an equilateral triangle, to construct the equilateral triangle.

23. Given the base and perimeter of an isosceles triangle, to construct the isosceles triangle.

24. Given two points A and B on the same side of a straight line CD and unequally distant from it, to find the shortest broken line that can be drawn from A to the line CD and thence to the point B.

BOOK III. PROPORTION, SIMILAR POLYGONS.

11. Proportion.

Find the missing term in the following proportions —

1. $6 : 9 = 18 : (\ )$.
2. $13 : 17 = 19 : (\ )$.
3. $3 : 10 = 37 : (\ )$.
4. $2\frac{1}{3} : 3\frac{1}{4} = 4\frac{1}{6} : (\ )$.
5. $19 : 9 = (\ ) : 79$.
6. $(\ ) : x + y = x^2 - xy + y^2 : x^3 + y^3$.
7. $x^2 - 5x + 6 : x^2 - 7x + 12 = (\ ) : x^2 - 6x + 8$.

Derive proportions from the following equations:

1. $ab = 91$.
2. $7x = 11y$.
3. $y^2 = xz$.
4. $x^2 - y^2 = m^2 - n^2$.
5. $a^3 + b^3 = c^3 - d^3$.
6. $57 = x^2 - y^2$.
7. $x^2 + 23x + 22 = a^2 + 2a + 1$.

Put the proportion $20 : 18 = 10 : 9$, in proportion:

1. By Alternation,
2. By Inversion,
3. By Composition,
4. By Division,
5. By Composition and Division.

Solve the equations:

1. $x + 2 : x + 3 = x + 9 : x + 11$.
2. $2x + 1 : 3x - 1 = 4x + 3 : 5x + 1$.
3. $x + a : x + b = x + c : x + d$.
4. $x : x + a = x + a + 1 : x + a + 2$.
5. $x^3 + 8 : x^3 - 8 = 9 : 7$.
6. $2x + 3 : 3x - 5 = x : x + 7$.
7. $3x + 2 : 7x - 1 = 5x + 1 : 10x + 2$.

12. Similar Triangles.

Tests of Similarity. Two triangles are similar if —

1. They are mutually equiangular.
2. Their homologous sides are proportional.
3. They have an angle of one equal to an angle of the other, and the sides including the equal angles are proportional.
4. The homologous sides are parallel.
5. The homologous sides are perpendicular.

13. Exercises. — Application of Tests.

1. The triangle DEF is formed by joining D, E, and F, the middle points respectively of the sides AB, AC, and BC of the triangle ABC. Prove DEF similar to ABC.

2. The corresponding triangles formed by drawing the homologous diagonals of two similar trapeziums, are similar.

3. Two isosceles triangles which have an angle of the one equal to a homologous angle of the other, are similar.

4. AB and CD are the bases of a trapezoid and O is the point of meeting of its diagonals; prove that the triangles AOB and COD are similar.

5. Two triangles formed by joining the extremities of two chords which intersect within the circle, are similar.

6. A circle is tangent internally to another circle at A; AB and AC, chords of the larger circle, meet the circumference of the smaller circle at D and E; join DE and BC and prove that the triangles ADE and ABC are similar.

7. A circle is tangent externally to another circle at O; two secants are drawn through O, the one meeting the circumferences at A and B, the other meeting them at C and D; join AC and BD, and prove that the triangles OAC and OBD are similar.

8. In the triangle ABC, AD is drawn perpendicular to the side BC, and BE is drawn perpendicular to the side AC; prove that the triangles ACD and BEC are similar.

9. In the triangle ABC, AB is 8 inches and AC 12 inches; a point D is taken 2 inches from A on AB, and another point E is taken 3 inches from A on AC; join DE and prove that the triangle ADE is similar to the triangle ABC.

14. Numerical Problems.

1. In the similar triangles ABC and $A'B'C'$, AB is 20, AC is 30, and $A'B'$ is 12; find $A'C'$.

2. In the same triangles, if AB is 3, BC is 17, and $B'C'$ is 19; find $A'B'$.

3. In the same triangles, if AB is 12, AC is 16, BC is 20, and $A'B'$ is 15; find the perimeter of $A'B'C'$.

In the triangle ABC, DE is drawn parallel to the base.

4. Given AD 10, BD 4, and AE 12; to find CE.

5. Given AB 42, AD 35, and EC 14; to find AE.

6. Given BC 20, DE 8, and AD 6; to find AB.

7. Given AE 30, EC 10, and BC 25; to find DE.

8. Given BC 10, DE 8, and BD 6; to find AD.

9. Given BC 20, DE 8, and the altitude of the triangle ABC 24; to find the altitude of the triangle ADE.

10. Given AD 14, AB 16, and the perimeter of the triangle ABC 32; to find the perimeter of the triangle ADE.

11. Given BC 12, DE 5, and the distance from A to DE 9; to find the distance from A to BC.

12. Given BC 15, DE 10, and the distance from DE to BC 4; to find the distance from A to BC.

15. Theorem.

If from the vertex of the right angle of a right triangle, a perpendicular be dropped to the hypothenuse,

The perpendicular is a mean proportional between the segments of the hypothenuse.

Each side about the right angle is a mean proportional between the hypothenuse and its adjacent segment.

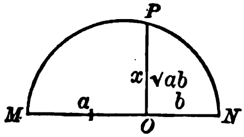
COR. *A perpendicular dropped from any point in the circumference of a circle to its diameter is a mean proportional between the segments of the diameter.*

In the following exercises, $h=$ the hypothenuse of a right triangle, $p=$ the perpendicular from the vertex of the right angle to the base, a and $b=$ the legs of the right triangle, and c and $d=$ the respective adjacent segments of the hypothenuse.

1. Given $c=6$; $d=24$, to find p.
2. Given $c=45$; $d=75$, to find p.
3. Given $c=m^2n$; $d=mn^2$, to find p.
4. Given $c=3$; $d=9$, to find p.
5. Given $p=8$; $c=4$, to find d.
6. Given $a=4$; $c=2$, to find d.
7. Given $a=5$; $c=3$, to find h.
8. Given $c=12$; $d=15$, to find a.
9. Given $p=6$; $h=20$, to find c and d.
10. Given $a=14$; $c=7$, to find d.

16. Problems in Construction.

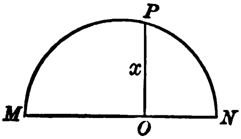
(1) Represent two finite straight lines by a and b, and construct $\sqrt{ab}$.



Construct a straight line MN equal to the sum of a and b. On MN as a diameter, describe the semi-circumference MPN. At O draw OP, (x), perpendicular to MN; then

$$a:x=x:b, \text{ (15) Cor.}$$
$$x^2=ab,$$
$$x=\sqrt{ab}.$$

(2) Represent one unit by a straight line and construct $\sqrt{2}$.



Construct a straight line MN equal to three of the chosen units. Describe on MN the semi-circumference MPN. At the extremity of the second measure draw OP perpendicular to MN; then

$$2:x=x:1, \text{ (15) Cor.}$$
$$x^2=2$$
$$x=\sqrt{2}$$

In the first six exercises below, make $a>b>c$. Construct

1. $\sqrt{ab}+\sqrt{bc}$
2. $a+\sqrt{ab}$
3. $a-\sqrt{ab}$
4. $\sqrt{a(b+c)}$
5. $\sqrt{b(a-c)}$
6. $\sqrt{(a+b)(c-b)}$
7. $\sqrt{3}$
8. $\sqrt{5}$
9. $\sqrt{6}$
10. $\sqrt{5}+\sqrt{3}$
11. $\sqrt{7}-\sqrt{2}$
12. $\sqrt{\frac{1}{2}}$

Divide a given straight line in the proportion:

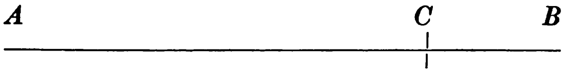
1. Of 5 and 6. 2. Of 1 and $\sqrt{3}$. 3. Of $\sqrt{2}$ and $\sqrt{3}$.
4. Of $\sqrt{6}+\sqrt{2}$ and $\sqrt{6}-\sqrt{2}$. 5. Of $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$.

Divide a given straight line so that the one segment shall be to the whole line

1. As 1 to 3.
2. As 1 to $\sqrt{2}$.
3. As $\sqrt{3}$ to $\sqrt{5}$.

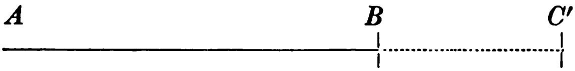
17. Definitions.

1. A finite straight line is divided internally in a given ratio when the point of division is within the line and the segments are in the given ratio.



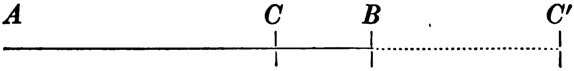
If the line AB is divided at C, so that AC is three times BC, the line is divided at C in the proportion of 3 to 1.

2. A finite straight line is divided externally in a given ratio when the point of division is without the line and the segments are in the given ratio.



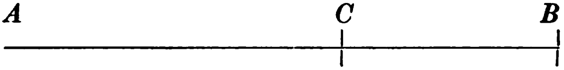
If the line AB be produced to C', so that AC' is three times BC', the line is divided externally in the ratio of 3 to 1.

3. A finite straight line is divided harmonically in a given ratio when it is divided externally and internally in the same ratio.



If the line AB be divided at C so that AC is to CB as 3 is to 1, and also divided at C' so that AC' is to BC' as 3 is to 1, the line is divided harmonically in the ratio of 3 to 1. The proportion may be written $AC:CB=AC':BC'$.

4. A line is divided in Extreme and Mean Ratio when the whole line is to the longer segment as the longer segment is to the shorter segment.



If the line AB is divided at C so that $AB:AC=AC:CB$, the line is divided in extreme and mean ratio.

---

1. Divide a given straight line internally in the ratio of 3 to 5; of 4 to 7; of $1\frac{1}{2}$ to $1\frac{1}{3}$; of $\sqrt{3}$ to $\sqrt{5}$.

2. Divide a given line externally in the ratio of 5 to 2; of $\sqrt{5}$ to $\sqrt{3}$; of $\sqrt{6}$ to $\sqrt{3}$.

3. To divide a given straight line harmonically in the ratio of 6 to 5; of 3 to $\sqrt{7}$; of $\sqrt{12}$ to $\sqrt{7}$; of two given straight lines.

4. A line 8 inches long is divided internally in extreme and mean ratio; find the length of each segment.

5. A line 10 inches long is divided internally in extreme and mean ratio; find the length of each segment.

6. The straight line a is divided internally in extreme and mean ratio; find an expression for the length of each segment.

7. Given the shorter segment of a line divided in the ratio of 2 to $\sqrt{3}$, to find the entire line.

8. Divide a given line internally in the proportion of $2+\sqrt{3}$ and $2-\sqrt{3}$.

BOOK IV. MEASUREMENTS OF SURFACES.

18. Theorem.

The area of a triangle is equal to one-half the product of its base by its altitude.

1. Given the base of a triangle 46 and its altitude 44; to find its area.

2. Given the base of a triangle $12\frac{1}{2}$ and its altitude $6\frac{1}{4}$; to find its area.

3. Given the area of a triangle 96 and its base 24; to find its altitude.

4. Given the area of a triangle 126 and its altitude 84; to find its base.

5. Divide a triangle into two equivalent parts by means of lines from the vertex to the base; into three equivalent parts; into five equivalent parts.

6. Prove the equivalence of the four triangles formed by drawing the diagonals of a parallelogram.

7. Prove the equivalence of the six triangles formed by drawing the medians of a triangle.

8. The perimeter of a polygon circumscribed about a circle is 49, and the radius of the circle is 6; find the area of the polygon.

9. The area of a circumscribed polygon is 64 and its perimeter is 16; find the radius of the inscribed circle.

10. The side of an equilateral triangle is m, and the radius of the inscribed circle is 6; find the area of the triangle.

19. Theorem.

The area of a trapezoid is equal to half the sum of its parallel sides by its altitude.

1. Given the parallel sides of a trapezoid 28 and 26, and its altitude 14; to find its area.

2. The parallel sides of a trapezoid are 40 and 14, and its area is 81; find its altitude.

3. The parallel sides of a trapezoid are 28 and 18, and its altitude is 6; find the area of each of the triangles formed by drawing a diagonal of the trapezoid.

4. To divide a given trapezoid into two equivalent trapezoids.

5. The parallel sides of a trapezoid are 20 and 14, and its altitude is 8; find the altitude of the triangle formed by producing the non-parallel sides of the trapezoid till they meet.

6. The parallel sides of a trapezoid are 48 and 42, and its altitude 12; find the altitudes of the two triangles formed as in 5.

7. The parallel sides of a trapezoid are a and b, and its altitude is c; if a is the longer side, find the altitudes of the two triangles formed as in 5.

8. The area of a triangle is 288; find the area of a trapezoid formed by passing a line through the middle points of its sides.

9. The base of a triangle is 30 and its area is 240; find the altitude of the trapezoid formed by passing a line through the middle points of the sides adjacent to the base.

10. The area of a triangle is m and its base is n; find the altitude of the trapezoid formed by passing a line

through the middle points of the sides adjacent to the base.

20. Theorem.

Similar polygons are to each other as the squares of their homologous sides.

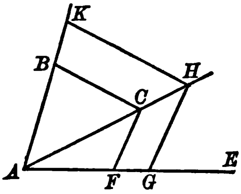
1. Find the ratio of two polygons whose sides are in the ratio 2 to 7.

2. Find the ratio of the homologous sides of two polygons whose ratio is 16 to 49; whose ratio is 12 to 18.

3. One side of a triangle is 5 and its area is 24; find the homologous side of a similar triangle one fourth as large.

21. Problems in Construction.

1. Given a triangle, to construct a similar triangle having twice its area.



Given the triangle ABC, to construct a triangle having double its area.

From any vertex as A, draw AE, making any convenient angle with a side as AC. Lay off on AE, AF and AG, respectively proportional to 1 and $\sqrt{2}$. Draw GH parallel to FC to meet AC produced at H. Draw HK parallel to BC to meet AB produced at K.

Triangles ABC and AKH are similar; therefore since $AC : AH = 1 : \sqrt{2}$; $ABC : AKH = 1 : 2$.

$\therefore$ AKH has double the area of ABC.

2. Given an equilateral triangle, to construct an equilateral triangle having four times its area.

3. Given a triangle, to construct a similar triangle three times as large.

4. Given a triangle, to construct a similar triangle that shall have to it the ratio of 3 to 2.

5. Given a trapezium, to construct a similar trapezium that shall have to it the ratio of 2 to 5.

6. Given a pentagon, to construct a similar pentagon that shall have one-half its area.

7. Construct two trapeziums that shall be to each other as 3 to 5.

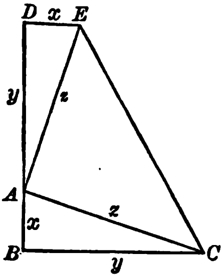
22. Theorem.

The square described on the hypothenuse of a right triangle is equal to the sum of the squares described on the other two sides.

This is called the Pythagorean Proposition from the tradition that it was first demonstrated by Pythagoras, about 550 B. C. The demonstration usually found in the text-books is supposed to have been discovered by Euclid — "the Father of Geometry" — who flourished in the third century before Christ.

The number of published demonstrations of this famous theorem is practically unlimited, and additional proofs are constantly appearing in educational and mathematical publications. In the library of the Iowa State Normal School the writer has found about two hundred different methods of proöf.

One of the best and most interesting of the published proofs is known as the Garfield demonstration. It was discovered by James A. Garfield when he was a student in Hiram College. At that time, original demonstration played an insignificant part in geometrical teaching; and the story goes that young Garfield came to his instructor with the claim that he had originated a new proof of the Pythagorean theorem. The teacher was skeptical, but promised to investigate his claim if he would write out his demonstration. He did so, and on the following day the teacher returned his manuscript, saying: "You have not only a good demonstration of the proposition — you have the best demonstration." Garfield's method of proof is given below. Whether his teacher's verdict can be verified or not, the proof is certainly one of the simplest and best.



Given the right triangle *ABC*, the sides of which are represented by x, y, and z.

To prove $x^2+y^2=z^2$.

Produce *BA* to *D* making *AD* equal to y. At *D* draw *DE* equal to x and perpendicular to *DA*. Draw *AE* and *EC*.

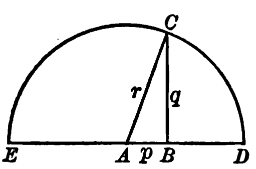
The triangle *DEA* is evidently equal to the triangle *ABC*, and the angle *EAC* is evidently a right angle.

The trapezoid *DECB* is composed of three right triangles *ABC*, *ADE*, and *EAC*. The area of the trapezoid is $\frac{1}{2}(x+y)(x+y)$, and the areas of the triangles are respectively $\frac{1}{2}xy$, $\frac{1}{2}xy$, and $\frac{1}{2}z^2$.

Therefore $\frac{1}{2}(x+y)(x+y)=\frac{1}{2}xy+\frac{1}{2}xy+\frac{1}{2}z^2$.

Clearing of fractions and reducing, $x^2+y^2=z^2$.

The following brief and simple demonstration, so far as the writer knows, has never before been printed.



Given the right triangle *ABC*, the sides of which are represented by p, q, and r.

To prove $p^2+q^2=r^2$.

With *A* as a center and r as a radius, describe the semicircle *DCE*. *CB* (q) is a mean proportional between the segments *EB* and *BD* of the diameter.

EB equals $r+p$, and *BD* equals $r-p$.

Therefore $(r+p)(r-p)=q^2$.
Multiplying, $r^2-p^2=q^2$.
Transposing, $r^2=p^2+q^2$.

Q. E. D.

23. Problems.

In the first fourteen of the following exercises, h designates the hypothenuse; b, the base; a, the altitude; p, the perimeter, and s, the area.

1. Express h in terms of a and b.

2. Express b in terms of h and a.

Given:

3. $b=x$; $a=2x$; to find h.

4. $h=10x$; $b=6x$; to find a.

5. $b=x+y$; $a=x-y$; to find h.

6. $h=x^2+y^2$; $a=2x-y^2$; to find b.

7. $h=\frac{m}{n}+\frac{n}{m}$; $b=\frac{m}{n}-\frac{n}{m}$; to find a.

8. $a=27$; $b=36$; to find h.

9. $h=95$; $b=76$; to find a.

10. $a+b=49$; $h=35$; to find a and b.

11. $h+a=24$; $b=12$; to find a.

12. $h-b=6$; $a=18$; to find h and b.

13. $p=48$; $s=96$; to find a, b, and h.

14. $h=40$; $s=384$; to find a, b, and h.

15. A pole 90 feet high is broken off, the trunk remaining on the stump; the top strikes the ground thirty feet from the foot of the pole; find the height of the stump.

16. The sides of an isosceles triangle are 10, 10, and 16; find its altitude. Find its area.

17. The sides of an isosceles triangle are 12, 12, and 8; find its altitude. Find its area.

18. The side of a square is 8; find its diagonal.

19. The side of a square is a; find its diagonal.

20. The side of an equilateral triangle is 6; find its altitude. Find its area.

21. Find the area of an equilateral triangle if its side is 8; is 10; is 12; is 20; is a.

22. The altitude of an equilateral triangle is 18; find its side.

23. The altitude of an equilateral triangle is a; find its side.

24. The sides of a triangle are 8, 10, and 12; find the length of the altitude from the vertex of the largest angle to the opposite side.

25. The altitude of an equilateral triangle is 30; find its area.

26. Two sides of a triangle are 15 and 15, and its altitude from the angle enclosed by those sides is 12; find its base and area.

27. The parallel sides of an isosceles trapezoid are 24 and 36, and its non-parallel sides are 10 each; find the area of the trapezoid.

28. The parallel sides of an isosceles trapezoid are 16 and 18, and the non-parallel sides are 6 each; find its area.

29. The longer diagonal of a rhombus is twice the shorter diagonal, and the side of the rhombus is 40; find its diagonals and its area.

30. The diagonal of a square is 24; find its area.

31. The diagonal of a square is d; find its area.

32. One angle of a right triangle is 30, and the side opposite this angle 12; find the area of the triangle.

33. One side of an isosceles right triangle is 20; find its hypothenuse.

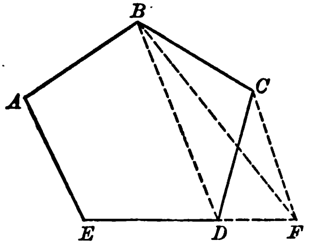
34. A and B are the sides of an obtuse triangle. A perpendicular is drawn from the vertex of the obtuse angle to the base dividing it into two segments, C and D. Show:

(*a*) That $A^2+D^2=B^2+C^2$.
(*b*) That $A+B:C+D::C-D:A-B$.

35. Represent three straight lines by A, B, and C, making $A>B>C$, and construct,

(*a*) A^2+B^2.
(*b*) A^2-B^2.
(*c*) $A^2+B^2+C^2$.
(*d*) $A^2+B^2-C^2$.

24. Problems in Construction.



1. To construct a quadrilateral equivalent to a given pentagon:

Given the pentagon $ABCDE$, to construct a quadrilateral equivalent to it. Draw the diagonal BD.

From the vertex C, draw CF parallel to BD, meeting ED, produced to F.

Triangles BCD and BDF are equivalent, having a common base BD and a common altitude — the distance between the parallel lines BD and CF.

The common part $ABDE+$the triangle BDF is equivalent to $ABDE+$the triangle BCD; that is, the quadrilateral $ABFE=$the pentagon $ABCDE$.

Q. E. F.

Remark: By a like process, from a polygon of any number of sides, an equivalent polygon of one fewer number of sides may be constructed; and, evidently by repeating the process a sufficient number of times, a triangle may be obtained equivalent in area to any polygon.

2. To construct a square equivalent to a given triangle.

Suggestion: Find a mean proportional between the base and half the altitude.

3. To construct a triangle equivalent to a given square.

4. To construct a square equivalent to a given parallelogram.

5. To construct a square equivalent to a given quadrilateral one of the angles of which is re-entrant.

6. To construct a triangle equivalent to a given polygon having the form of a five-pointed star.

7. To construct a square that shall be two-thirds of a given square.

8. To construct a square equivalent to the sum of two given triangles.

25. Theorem.

The square described on the side of a triangle opposite an acute angle is equal to the sum of the squares described on the other two sides diminished by twice the product of one of those sides by the projection of the other side upon it.

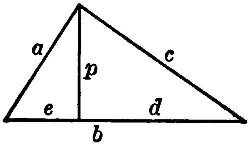
Problem.

To find the area of a triangle, when the three sides are given.

Rule.

From half the sum of the three sides, take each side separately, and extract the square root of the continued product of the half-sum and the three remainders.

26. Demonstration.



In the figure, a, b, and c represent the three sides of a triangle; p represents the perpendicular from the vertex to the base; e and d represent the segments of the base made by the perpendicular.

$$a^2=b^2+c^2-2\,bd \ (25)$$

$$d=\frac{b^2+c^2-a^2}{2\,b}$$

$$p^2=c^2-d^2=c^2-\frac{(b^2+c^2-a^2)^2}{4\,b^2}$$

$$=\frac{4\,b^2\,c^2-(b^2+c^2-a^2)^2}{4\,b^2}$$

$$=\frac{(2\,bc+b^2+c^2-a^2)\ (2\,bc-b^2-c^2+a^2)}{4\,b^2}$$

$$=\frac{[(b^2+2bc+c^2)-a^2]\ [a^2-(b^2-2bc+c^2)]}{4\,b^2}$$

$$=\frac{(a+b+c)\ (b+c-a)\ (a-b+c)\ (a+b-c)}{4\,b^2}$$

$$p=\frac{1}{2\,b}\sqrt{(a+b+c)\ (b+c-a)\ (a-b+c)\ (a+b-c)}$$

Multiplying both members by $\frac{b}{2}$ to obtain area of triangle.

$$\frac{pb}{2}=\frac{1}{4}\sqrt{(a+b+c)\ (b+c-a)\ (a-b+c)\ (a+b-c)}$$

Placing the denominator under the radical and factoring.

$$\frac{pb}{2}=\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}$$

put $\frac{a+b+c}{2}$ = half the sum of the sides.

$\frac{a+b+c}{2}-a=\frac{b+c-a}{2}$ or half the sum minus a.

In like manner $\frac{a-b+c}{2}$ = the half sum minus b, and $\frac{a+b-c}{2}$ = the half sum minus c.

Hence the Rule.

Remark: By placing $\frac{a+b+c}{2}=s$, the formula may be simplified to the form $=\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$.

27. Applications.

Given the sides of a triangle 9, 10, and 11; to find its area.

Substituting in the formula, since $s=15$, $s-a=6$, $s-b=5$ and $s-c=4$; we have $\sqrt{15\times6\times5\times4}$.

It may be readily seen by inspection that the numbers 5, 3, and 2 each appear twice under the radical as a factor.

Simplifying, $\sqrt{15\times6\times5\times4}=30\sqrt{2}$.

Given the sides of a triangle 7, 9, and 11; to find its area.

Substituting in formula, $\sqrt{\frac{27}{2}\times\frac{13}{2}\times\frac{9}{2}\times\frac{5}{2}}$

Here 9 occurs twice as a factor in the numerators and 2 four times as a factor in the denominators, giving the simplified result $\frac{9}{4}\sqrt{195}$.

With a little practice, exercises like the above involving small numbers may be performed mentally by the pupil.

While the foregoing rule is applicable to the solution of every form of plane triangle, the areas of equilateral and isosceles triangles are more readily obtained by employing the Pythagorean theorem to find the altitude, and multiplying it by half the base.

28. Problems.

1. Find the area of a triangle whose sides are 6, 8, and 10; 9, 12, and 15; 7, 10, and 11; 13, 14, and 15; 14, 15, and 16; 16, 18, and 20; 17, 19, and 21; 23, 26, and 29; 45, 57, and 63; 67, 77, and 90; 123, 124, and 125; 300, 600, and 700.

2. The sides of a quadrilateral are $AB=3$, $BC=4$, $CD=16$, and $DA=15$, and the angle B is a right angle; find the area of the quadrilateral.

3. In the quadrilateral $ABCD$, $AB=8$, $BC=6$, $CD=14$, $DA=16$, and the diagonal $AC=10$; find the area of the quadrilateral.

4. In the pentagon $ABCDE$, $AB=12$, $BC=13$, $CD=15$, $DE=16$, $EA=18$, the diagonal $AC=20$, and the diagonal $AD=22$; find the area of the pentagon.

5. The sides of a triangle are 12, 16, and 20; find altitude from the vertex of the largest angle to the base.

6. The sides of a triangle are AB 18, AC 20, and BC 22; find the altitude from the vertex A to the base.

7. Find the area of an isosceles triangle whose sides are 20, 20, and 30. Solve by the method of Art. 18, and verify the result by employing the method of Art. 27.

8. Prove by the method of Art. 27, that the area of an equilateral triangle whose side is a is $\frac{a^2}{4}\sqrt{3}$.

9. The sides of a triangle are 30, 32, and 34; find the radius of the inscribed circle. (See Art 18, Prob. 9.)

BOOK V. REGULAR POLYGONS AND CIRCLES.

29. Exercises.

1. How many degrees in one of the central angles of a regular polygon of 3 sides? of 11 sides? of n sides?

2. How many degrees in one of the equal angles of a regular polygon of 5 sides? of 7 sides? of n sides?

3. How many degrees in the angle included by the radius and apothem of a regular polygon of 10 sides? of 12 sides? of n sides?

4. The side of a regular hexagon is 4; find its area.

5. The side of a regular hexagon is n; find its area.

6. The apothem of a regular hexagon is $4\sqrt{3}$; find its area.

7. The side of a regular octagon is 8; find its area.

8. The side of a regular octagon is 12; find its area.

9. The side of a regular octagon is n; find its area.

30. The Regular Hexagon.

Given the regular hexagon $ABCDEF$:

1. Divide it into two isosceles trapezoids.

2. Divide it into three equal rhombuses.

3. Join AE and BD; what form of quadrilateral is $ABDE$?

4. If FC cut AE and BD at I and J, is the figure $ABJI$ a square?

5. Join AC, AE, and CE; is the triangle formed equilateral?

6. Join BD, BF, and DF, intersecting AE, AC, and EC at m, n, p, q, r, and s; is the hexagon $mnpqrs$ regular?

7. Is *FB* trisected by the points *m* and *n*?

8. Test the equality and equivalence of the triangles *FAM*, *MAN*, and *ANB*.

9. Find the ratio of the hexagon *ABCDEF* to the hexagon *mnpqrs*.

10. Given a circle, to find the ratio of the regular hexagon circumscribed about the circle to the regular hexagon inscribed in it.

11. Inscribe a regular hexagon in a given equilateral triangle.

12. Given a regular hexagon to find the ratio of the circumference of the circle circumscribed about it, to the circumference of the circle inscribed in it.

13. Given a regular hexagon, to find the ratio of the circumscribed circle to the inscribed circle.

14. About a central circle *A*, place six other circles each with the same radius as *A*, so that each shall touch *A* and also touch two other circles; draw common external tangents to each pair of outer circles and prove that the hexagon formed is regular.

31. Exercises with Polygons and Circles.

Given the radius of the circumscribed circle 10, to find the area of the inscribed.

1. Equilateral triangle.
2. Square.
3. Regular hexagon.
4. Regular octagon.
5. Regular dodecagon.
6. Find the circumference of a circle whose radius is 18; find its area.

7. The circumference of a circle is 20; find its area.

8. The area of a circle is 6283.2; find its circumference.

9. Find the area of the largest wheel that can be cut from a board in the form of an equilateral triangle, if each side of the board is 20 inches.

10. The perimeter of a triangle is 50, and its area is 100; find the area of the inscribed circle.

11. If from any point in a regular polygon of m sides, perpendiculars be drawn to the sides, the sum of the perpendiculars is m times the radius of the inscribed circle.

12. In the inscribed hexagon $ABCDEF$, prove angle $A+$angle $C+$angle $E=$angle $B+$angle $D+$angle F.

13. Given three equal mutually tangent circles whose radius is 10, to find the area of the space without the circles, but included by the circumferences.

14. Find the space included, as in 13, if the radius of the equal circles is 12.

15. By means of concentric circumferences, to divide a given circle: (*a*) Into two equivalent parts. (*b*) Into three equivalent parts. (*c*) Into four equivalent parts.

16. To draw three circles in an equilateral triangle, so that each may touch two circles and also touch two sides of the triangle.

17. To draw three equal mutually tangent circles within a circle, so that each may touch its circumference.

18. To describe about a given circumference four equal circles, each of which shall touch the given circumference and the circumferences of the two remaining circles: (*a*) Within the circumference. (*b*) Without the circumference.

19. Find the surfaces included between two concentric circumferences, the radii of which are 6 and 8.

BOOK VI. PLANES.

32. Symmetry.

Symmetry is such an arrangement of two geometrical magnitudes or of the parts of a geometrical magnitude with reference to a center that any point in the one has a corresponding point in the other on the opposite side of the center and equidistant with it from the center. A surface which can be divided into two equal parts which are thus related is called a symmetrical surface. A solid which can be so divided is called a symmetrical solid.

I. With reference to a point:

1. Two points are symmetrical, if the straight line connecting them is bisected by the point.
2. Two lines are symmetrical, if every straight line terminated by the lines and passing through the point is bisected by it.
3. Two planes are symmetrical, if every straight line terminated by the planes and passing through the point is bisected by it.

II. With reference to an axis:

1. Two points are symmetrical, if the straight line connecting them is perpendicular to the axis and is bisected by it.
2. Two lines are symmetrical, if every straight line terminated by the lines and drawn perpendicular to the axis is bisected by it.
3. Two planes are symmetrical, if every straight line terminated by the planes and drawn perpendicular to the axis is bisected by it.

III. With reference to a plane:

1. Two points are symmetrical if the straight line connecting them is perpendicular to the plane and is bisected by it.
2. Two lines are symmetrical, if every straight line terminated by the lines and drawn perpendicular to the plane is bisected by it.
3. Two planes are symmetrical, if every straight line terminated by them and drawn perpendicular to the plane is bisected by it.

Let the pupil represent by diagrams the nine different forms of symmetry described above.

A Symmetrical Solid is a solid every point of which has a homologous point in the figure on the opposite side of the center of symmetry and equidistant with it from the center.

Perfect Symmetry is symmetry with reference to a point.

Relative Symmetry is symmetry with reference to an axis or a plane.

If the form of a symmetrical figure remains unchanged by being turned through half a revolution about its center, it is said to have two-fold symmetry. If the coincidence occurs after one-third of a revolution, it is said to have three-fold symmetry; if after one-fourth of a revolution, four-fold symmetry, and so on; in general, if its form remains unchanged after one-*n*th of a revolution, it is said to have *n*-fold symmetry.

What kind of symmetry has an equilateral triangle? a regular heptagon? the lily? the lily of the valley? the apple blossom? the mustard flower? the starfish (approximately)? a carriage wheel?

Remark. Both nature and art furnish an infinite variety of illustrations of the various forms of symmetry. To these, the pupils' minds may be very profitably directed.

33. Loci in Space.

In solid geometry, the locus of all points possessing a common property must be considered, not only with reference to a plane, but with reference to space. Thus the locus of a point in space equidistant from two given points, is the plane which is perpendicular to, and bisects, the straight line conecting the points.

1. What is the locus in space of a point at a given distance from a given plane?

2. What is the locus of a point equidistant from three given points?

3. What is the locus in space of a point equidistant:
   (*a*) From two given parallel straight lines?
   (*b*) From two given intersecting straight lines?
   (*c*) From two given parallel planes?
   (*d*) From two given intersecting planes.

4. Find the locus of a point equidistant from two points and also equidistant from two parallel lines.

5. Find the locus of a point equidistant from two points and also equidistant from two given intersecting planes.

6. Find the locus of a point equidistant from the three faces of a trihedral angle.

7. Find the locus of a point equidistant from the three edges of a trihedral angle.

8. Given four points not all in the same plane, to find a fifth point equidistant from the four.

BOOK VII. SOLIDS.

34. Abbreviations Employed in Formulæ.

A. Altitude.
B. Base.
b. Upper Base.
H. Slant Height.
P. Perimeter of Base, Perimeter of Lower Base.
p. Perimeter of Upper Base.
R. Radius, Radius of Base, Radius of Lower Base.
r. Radius of Upper Base.
S. Surface.
V. Volume.

35. Theorem.

The lateral surface of a right prism is equal to the product of the perimeter of the base by the altitude.

Formula: $S = PA$.

1. The lateral edge of a right prism is 17 and its basal edges are 7, 11, 13, and 15; find its lateral surface.

2. The diagonal of a cube is $12\sqrt{3}$; find its surface.

3. Each edge of a triangular right prism is 8, and its lateral surface is 20; find its altitude.

4. The base of a right prism is an equilateral triangle, the apothem of which is 10, the altitude of the prism is 16; find its lateral surface.

5. Find the entire surface of a right prism, the base of which is a regular hexagon, and the altitude of which is equal to the apothem of the base, the basal edges being 6.

6. The breadth of a rectangular parallelopiped is one more than the height, and the length is one more than the breadth, and its entire surface is 214; find its dimensions.

7. The breadth of a rectangular parallelopiped is twice the height, and the length is three times the height; its entire surface is 88; find its dimensions.

8. One edge of a prism is 6, and its entire surface is 11; find the surface of a similar prism whose homologous edge is 12.

9. The base of a right prism is a right triangle, and its altitude is equal to the hypothenuse of the triangle forming the base, the legs of which are 8 and 6; find the entire surface of the prism.

36. Theorem.

The volume of a right prism is equal to the product of its base by the altitude.

Formula: $V = BA$.

1. The surface of a cube is 12; find its volume.

2. The volume of a cube is 12; find its surface.

3. The basal edges of a triangular prism are 5, 5, and 8, and its altitude is 23; find its volume.

4. Each edge of a triangular right prism is 6; find its volume.

5. The basal edge of a regular triangular right prism is 6, and its lateral edge is 8; find its volume.

6. The altitude of a right prism is 14, and its base is an isosceles trapezoid the sides of which are 18, 12, 6, and 6; find the volume of the prism.

7. The basal edges of a prism are 10, 12, and 14, and its altitude is 7; find its volume.

8. The basal edges of a prism are 13, 16, and 17, and its altitude is 2; find its volume.

9. Find the volume of a regular hexagonal prism, its basal edge 10 and its lateral edge 16.

10. Find the volume of a regular hexagonal prism, its basal edge a and its lateral edge b.

11. Each edge of a triangular prism is equal to the edge of a cube; find the ratio of the volume of the prism to the volume of the cube.

12. The diagonal of a cube is 12; find its volume.

13. Derive a formula for the diagonal of a rectangular parallelopiped in terms of its three dimensions.

14. A room is 30 ft. long, 20 ft. wide, and 12 ft. high; find the shortest route by which a fly may walk from a lower corner of a room to the diagonally opposite upper corner.

37. Theorem.

The lateral surface of a regular pyramid is equal to one-half the product of the perimeter of the base by the slant height.

Formula: $S = \frac{1}{2} PH.$

1. The basal edge of a regular heptagonal pyramid is 9, and its slant height is 20; find its lateral surface.

2. The basal edge of a regular triangular pyramid is 8, and its lateral edge is 10; find its lateral surface.

3. Each edge of a triangular pyramid is 18; find its entire surface.

4. The entire surface of a triangular pyramid, all the edges of which are equal, is $64\sqrt{3}$; find its edge.

5. The lateral edge of a square pyramid is 6, and its basal edge is 8; find its lateral surface.

6. The slant height of a regular square pyramid is 8, and its lateral surface is 192; find its entire surface.

7. The slant height of a regular square pyramid is 10, and its lateral surface is 224; find its entire surface.

8. A lateral edge of a pyramid is 9; find the homologous lateral edge of a similar pyramid whose surface is four times as great.

9. The base of a regular square pyramid is 64, and its altitude is 7; find its lateral surface.

10. The base of a regular square pyramid is 100, and its altitude is 4; find its entire surface.

38. Theorem.

The volume of any pyramid is equal to one-third of the product of the area of its base by its altitude.

Formula: $V = \frac{1}{3} AB$.

1. Find the volume of a pyramid, base 20, altitude 21.

2. Find the volume of a pyramid, basal edges 10, 11, and 12, altitude 6.

3. Find the volume of a regular tetrahedron if each of its edges is 8.

4. Find the volume of a regular octahedron if each of its edges is 8.

5. Find the volume of a hexagonal pyramid if each of its lateral edges is 30, and its altitude is 24.

6. Find the volume of a triangular pyramid, each of its basal edges 10, and each of its lateral edges 16.

7. Prove that the altitudes of a regular tetrahedron intersect in a point cutting off one-fourth of each altitude.

39. Theorem.

The lateral surface of a frustum of a regular pyramid is equal to one-half the product of the sum of the perimeters of the upper and lower base by the slant height.

Formula: $S=\frac{1}{2}H(P+p)$.

1. Find the lateral surface of a frustum of a regular octagonal pyramid, given the edge of the lower base 13, the edge of the upper base 7, and the slant height 22.

2. Given a frustum of a regular pentagonal pyramid, the edge of the lower base 56, the edge of the upper base 8, and the lateral edge 40; to find the lateral surface.

3. Given the frustum of a regular square pyramid, the edge of the lower base 32, the edge of the upper base 20, and the lateral edge 36; to find the lateral surface.

4. One of the lateral faces of a frustum of a regular pyramid is 88, and the corresponding basal edges are 22 and 2; find the slant height.

5. The lateral surface of a frustum of a regular pyramid is 380, its slant height is 20, and the perimeter of its upper base is 8; find the perimeter of its lower base.

40. Theorem.

The volume of a frustum of a pyramid is equal to one-third of its altitude multiplied by the sum of its upper base, its lower base, and a mean proportional between its bases.

Formula: $V=\frac{1}{3}A(B+b+\sqrt{Bb})$.

1. Each edge of the lower base of a frustum of a square pyramid is 9, each edge of its lower base is 7, and its altitude 21; find its volume.

2. Each edge of the lower base of a frustum of a square pyramid is 38, each edge of the upper base is 37, and its altitude is 36; find its volume.

3. The edges of the lower base of a frustum of a triangular pyramid are 12, 16, and 20, the corresponding edges of the upper bases are 9, 12, and 15, its altitude is 51; find the volume of the frustum.

4. The lower base of a frustum of a pyramid is 9, its altitude is 30, and its volume is 190; find its upper base.

5. How many board feet in a tapering stick of timber, two feet square at one end, one foot square at the other, and 32 feet long?

6. The altitude of a pyramid is 6; a plane is passed parallel to the base, dividing the pyramid into two equivalent parts; find the altitude of each part.

Suggestion: Similar solids are to each other as the cubes of their like dimensions.

7. The edge of the lower base of a square frustum is 20, the edge of the upper base is 16, and the slant height of the frustum is 18; find its volume.

41. Theorems.

The lateral surface of a cylinder of revolution is equal to the circumference of the base by the altitude.

Formula: $S = 2\pi AR$.

The volume of a cylinder of revolution is equal to the product of its base by its altitude.

Formula: $V = \pi AR^2$.

1. The circumference of the base of a cylinder of revolution is 12, and its altitude is 8; find its surface; find its volume.

2. The radius of the base of a cylinder of revolution is 8, and its altitude is 12; find its lateral surface; find its volume.

3. Find the volume of a cylinder if the radius of its base is 16, and its altitude is equal to the diameter of its base.

4. Given the radius of the base of a cylinder 43, and its altitude 17; to find its volume.

5. What length of wire one-eighth of an inch in diameter can be drawn out from a cubic foot of copper?

6. The diameter of a hollow cylinder whose altitude is 6, is 8, and the radius of the hollow part of the cylinder is 3; find the volume of the solid part.

7. Find the volume of a cylinder inscribed in a regular triangular prism whose basal edge is 14, and whose lateral edge is 12.

8. Find the volume of the cylinder circumscribed about the prism described in 7.

9. Find the volume of the cylinder circumscribed about a cube whose edge is 50.

42. Theorems.

The lateral surface of a cone of revolution is equal to one-half the product of the circumference of the base by the slant height.

Formula: $S = \pi RH$.

The volume of a cone is equal to the product of one-third of its base by its altitude.

Formula: $V = \frac{1}{3}\pi AR^2$.

1. The radius of the base of a cone of revolution is 10, and its slant height 20; find its lateral surface; find its volume.

2. The radius of the base of a cone of revolution is 20, and its altitude 15; find the lateral surface of the cone; find its volume.

3. Find the volume of a cone inscribed in a regular hexagonal pyramid whose lateral edge is 12, and whose basal edge is 4.

4. The lateral surface of a cone of revolution is 200 π, and the radius of its basis is 10; find the volume of the cone.

43. Theorems.

The lateral surface of a frustum of a cone of revolution is equal to one-half of the product of the slant height by the sum of the circumferences of the bases.

Formula: $S = \pi H (R + r)$.

The volume of a frustum of a cone is equal to one-third of the product of the altitude by the sum of the lower base, the upper base, and a mean proportional between the bases.

Formula: $\frac{1}{3} \pi A (R^2 + r^2 + Rr)$.

1. Given a frustum of a right cone, the radius of its lower base 6, the radius of its upper base 4, and its slant height 24; to find its lateral surface.

2. The volume of a frustum of a cone is 3380, the radii of its bases are 8 and 7; find its altitude.

3. The volume of a frustum of a cone is 10820 π; the radius of its lower base is 12, and its altitude is 90; find the radius of the upper base.

44. Theorems.

The surface of a sphere is equal to four great circles.

Formula: $S = 4 \pi R^2$.

The volume of a sphere is equal to the product of its surface by one-third of its radius.

Formula: $S=\frac{4}{3}\pi R^3$.

1. The radius of a sphere is 14; find its surface; find its volume.

2. The surface of a sphere is 1000; find its radius.

3. The surface of a sphere is 125664; find its radius.

4. The volume of a sphere is 113.0276; find its radius.

5. Find the volume of a sphere inscribed in a cube whose edge is 10.

6. Given a sphere whose radius is 12; to find the edge of the inscribed cube.

7. The ratio of the volumes of two spheres is 2:3; the radius of the first is 10; find the radius of the second.

8. A cylinder is circumscribed about a sphere; find the ratio:

   (*a*) Of the lateral surface of the cylinder to the surface of the sphere.

   (*b*) Of the entire surface of the cylinder to the surface of the sphere.

   (*c*) Of the volume of the cylinder to the volume of the sphere.

9. A cube is circumscribed about a sphere; find the ratio:

   (*a*) Of the surface of the cube to the surface of the sphere.

   (*b*) Of the volume of the cube to the volume of the sphere.

10. A hollow ball is ten feet in diameter; find the volume of the solid portion if it has a uniform thickness of two feet.

45. Miscellaneous Exercises.

1. Given the side of an equilateral triangle 40; to find its area.

2. Given the side of a regular hexagon 32; to find its area.

3. Given the side of a regular octagon 24; to find its area.

4. Given the apothem of a regular hexagon 8; to find its perimeter.

5. Given the diagonals of a rhombus 8 and 10; to find its perimeter.

6. Given the apothem of a regular octagon 12; to find the perimeter of the circumscribed square.

7. How many square yards of plastering in the walls and ceiling of a room 20 ft. long, 16 ft. wide, and 12 ft. high?

8. The sides of a triangle are 24, 28, and 40 ft.; find the lengths of the segments of the sides made by the points of tangency of the inscribed circle.

9. The altitude of a pyramid is 10; find the altitude of a similar pyramid whose volume is three times as great.

10. A plane passed parallel to the base of a pyramid whose altitude is 20, bisects the pyramid; find the altitude of the pyramid intercepted by the plane.

11. Three points — A, B, and C — are in a straight line; the distance from A to C is 16 inches, and B is equidistant between A and C; a fourth point D is 8 inches distant from both B and C; find the distance from A to D.

12. Find the altitude of a regular tetrahedon whose entire surface is $64\sqrt{3}$.

13. How many board feet in a cord of lumber?

14. Find the area of an isosceles right triangle whose hypothenuse is 20.

15. A hollow iron ball is two feet in diameter, the hollow part is one foot in diameter; find the volume of the solid portion of the ball.

16. Two chords on opposite sides of the center of a circle are 14 feet apart, their lengths are 12 feet and 16 feet respectively; what is the diameter of the circle?

17. The area of an equilateral triangle is $25\sqrt{3}$; find its side.

18. A pole 120 feet high, standing on a level plane, breaks off, leaving the broken piece still attached to the stump, and the top strikes the ground a certain distance from the pole; but if it had broken 8 feet higher, the top would have struck the ground 20 feet nearer the foot of the pole; find the height of the stump.

19. A horse is tethered to the corner of a rectangular building situated in the midst of a pasture by means of a hook placed four feet from the ground and a rope forty feet in length; if the building is more than forty feet square, find the number of square feet over which the horse can graze.

20. The diameter of a cylindrical cistern is 7 ft., it contains 3100 cu. in.; find its depth.

21. Find the ratio of an equilateral triangle, inscribed in a circle, to the equilateral triangle circumscribed about the same circle.

22. The diagonals of a quadrilateral intersect at right angles, so that the segments of one are 7 and 11, and of the other 9 and 12; find the area of the quadrilateral.

23. The sides of a triangle are 12, 15, and 18; find the length of the altitude drawn to the longest side.

Suggestion: The altitude of a triangle is equal to the area divided by half the base.

24. What is the length of the diagonal of a rectangle that is twice as long as it is wide, and has an area 2450 sq. ft.?

25. The outside dimensions of a bin built of two-inch lumber are 10 ft. 10 in., 7 ft. 7 in., and 4 ft. 4 in.; find the number of board feet of lumber required for its construction.

26. A square garden contains a hundred square feet, two paths whose courses are perpendicular to the sides and to each other intersect the garden and occupy one-half its area; find the width of the paths.

27. The perimeter of a rectangle is 180, and its area is 2000; find length and breadth.

28. Find the volume of a sphere circumscribed about a cube whose edge is 6.

29. Two men own a grindstone 18 inches in diameter. Allowing nothing for waste at the aperture, what part of the radius must each grind away in order to have an equal share of the stone?

30. Find the area of a circle inscribed in a triangle the sides of which are 35, 40, and 45.

Suggestion: The area of a circumscribed polygon is equal to one half its perimeter multiplied by the radius of the inscribed circle.

31. Find the surface of the largest wheel that can be cut from a triangular board whose edges are 20 in., 25 in., and 30 in.

32. If the earth's diameter is 8000 miles:

(*a*) Find its surface in square miles.
(*b*) Find its volume in cubic miles.

33. Given a sphere whose radius is 7; to find the volume of a spherical wedge formed by the meeting of two great circles which intersect at an angle of 60°.

34. The depth of a hemispherical vessel is 14 inches; find its capacity in liquid gallons.

35. Find the volume of the earth's atmosphere, if its depth is 50 miles, and the earth's radius is 4000 miles.

36. The radii of the bases of a frustum of a cone are 18 and 14, and the slant height of the frustum is 18; find its volume.

37. Produce the diameter AB of a circle to C; from C, with a radius equal to the radius of the circle, strike an arc cutting the circumference at D; draw CD and produce it till it meets the circumference again at E; prove that arc BD is one-third of arc AE.

38. Draw a radius through the middle point of the side of an inscribed equilateral triangle, and prove that the radius is bisected by the side.

39. In a circle whose radius is 12, find the area of the smaller segment intercepted by a chord perpendicular to the radius at its middle point.

40. Two vertical poles, a and c, are respectively 12 feet and 44 feet in height; a third pole, b, stands in a straight line between them and 6 feet from a; the distance from the top of a to the top of b is 10 feet, and the distance from the top of c to the top of b is 50 feet. Find the distance from the top of a to the top of c.

41. Find the area of a square inscribed in an isosceles triangle, if its base is 10 and its altitude 12.

Suggestion: One side of the square is the base of a triangle similar to the given triangle.

42. The sides of the lower base of a frustum of a triangular pyramid are 40, 48, and 56; the longest side of the upper base is 14; the altitude of the frustum is 17; find its volume.

Table of Formulæ.

1.	Circle	Circumference $= 2\pi R$. Area $= \pi R^2$.
2.	Prism	$S = PA$. $V = BA$.
	Cylinder	$S = 2\pi AR$. $V = \pi AR^2$.
3.	Pyramid	$S = \frac{1}{2} PH$. $V = \frac{1}{3} BA$.
	Cone	$S = \pi RH$. $V = \frac{1}{3} \pi R^2 A$.
4.	Frustum of a Pyramid	$S = \frac{1}{2} (P + p) H$. $V = \frac{1}{3} A (B + b + \sqrt{Bb})$.
	Frustum of a Cone	$S = \pi H (R + r)$. $V = \frac{1}{3} \pi A (R^2 + r^2 + Rr)$.
5.	Sphere	$S = 4\pi R^2$. $V = \frac{4}{3} \pi R^3$.

CHAPTER II

Original Demonstration.

(46) Geometry is a deductive science; that is, its general method of investigation is from the known to the unknown; its conclusions are reached either from self-evident truths or from truths previously established. The data for all geometrical reasoning are Definitions, Axioms, and Theorems.

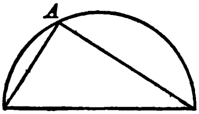
(47) A Theorem consists of two parts, the Hypothesis and the Conclusion. The Hypothesis is that which is given or supposed to be true. The Conclusion is that which is to be proved as a consequence of the hypothesis.

The hypothesis is generally introduced by the conjunction *if*. In the theorem, "If an angle is inscribed in a semi-circle, it is a right angle;" "If an angle is inscribed in a semi-circle," is the hypothesis: "It is a right angle," is the conclusion.

(48) The Converse of a Theorem is formed by making its hypothesis the conclusion, and its conclusion the hypothesis of another proposition. In the example given above, "If an angle is a right angle, it can be inscribed in a semi-circle," is converse of the theorem stated. The Converse of a Theorem may or may not be true. It is a proposition requiring demonstration.

Methods of Demonstration.

(49) 1. *The Direct Method.*—This consists in reasoning directly from categorical known premises to a sought conclusion; thus, all angles inscribed in a semi-circle are right angles.

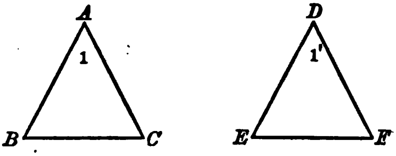


The angle A is inscribed in a semi-circle.

The angle A is a right angle.

(50) 2. *The Method of Superposition.*—This consists in placing one magnitude upon another to test the equality or similarity of its parts. It is illustrated in the following demonstration:

If two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other, the triangles are equal.



Given the triangles ABC and DEF, having, respectively, the side AB = the side DE; the side AC = the side DF, and the $<1 = <1'$.

To prove $\triangle ABC = \triangle DEF$.

Place $\triangle DEF$ upon $\triangle ABC$, so that $<1'$ shall fall on <1. Since $<1' = <1$, DE will lie on AB, and since $DE = AB$, the point E will fall on the point B. Since $DF = AC$, in like manner DF will lie on the line AC, and the point F will fall on the point C.

The lines EF and BC coincide, since their extremities are in the same points.

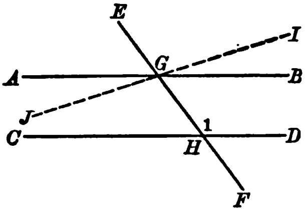
$\therefore$ the two triangles coincide throughout their whole extent and are equal.

Q. E. D.

(51) 3. *The Reductio ad Absurdum.*—This consists in assuming that the contradictory of the hypothesis is true, and reasoning by correct principles on the false assumption until an absurd conclusion is reached. Since the reasoning is correct and the conclusion is false, it follows

that the error is in the assumption, and that its contradictory — the original proposition — is true. This method is usually employed to prove the converse of a previously demonstrated proposition.

Illustration. — If two straight lines be cut by a transversal, making the alternate interior angles equal, the lines are parallel.



Given the straight lines AB and CD cut by the transversal EF, making $< AGH = < 1$.

To prove that AB is parallel to CD.

Suppose AB not parallel to CD, and draw IJ through G, parallel to CD.

Since IJ is parallel to CD,

$< JGH = < 1$. Alt.-int. $< S$.

But $< AGH = < 1$. Hypothesis.

$\therefore < AGH = < JGH$, since each is equal to < 1.

But this is impossible, since a part cannot equal the whole.

$\therefore$ IJ and AB must coincide, and AB is parallel to CD.

Q. E. D.

(52) 4. *The Exhaustive Method.* — This is a form of the reductio ad absurdum. It consists in proving that one of a series of propositions — one and only one of which can be true — is true, by showing that all the others are false. Thus of any two angles, A and B, one of the following propositions must be true: A is less than B, A is equal to B, A is greater than B. If it can be demonstrated that A is neither less than B, nor equal to B, the third proposition, A is greater than B, is true.

The Form of the Demonstration.

(53) The formal demonstration of a proposition, whether oral or written, consists of four distinct steps.

1. The Statement, consisting of the Hypothesis and Conclusion. It should be given by the student from memory in the language of the text-book.

2. The Restatement from the figure. (1) What is given, or the Hypothesis. (2) What is to be proved.

3. The additional lines, not required by the hypothesis, but essential to the demonstration. These should be either dotted lines, or drawn in a different color from that of those required by the Hypothesis.

4. The Demonstration.

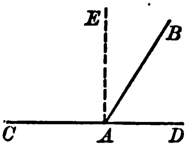
(54) These successive steps are shown in the mode below:

(1) *The Statement.*

(Hypothesis.) If one straight line meets another;

(Conclusion.) The sum of the two adjacent angles formed is equal to two right angles.

(2) *The Construction.*



Given the straight line AB meeting the straight CD at A.

To prove angle CAB + angle BAD equal to two right angles.

(3) *Additional Lines.*

At A draw EA perpendicular to CD.

(4) *The Demonstration.*

$<EAC + <EAD = 2$ rt. <s, since each is a right angle.

$<BAC+<BAD=<EAC+<EAD$, since each pair of angles occupies the same angular magnitude.

$<BAC+<BAD=2$ rt. $<$s. Things equal to the same thing are equal to each other. Q. E. D.

(55) A Problem is a question proposed for solution. A Problem in Geometry requires the construction of a figure in accordance with certain given conditions; as, To bisect a given straight line.

(56) Steps in the solution of a problem.

1. The Statement.

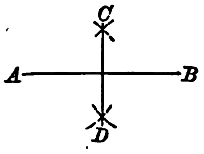
2. The figure, with statement of what is given, and of what is required.

3. The Construction.

4. The proof, — the principle or principles involved in the construction.

(59) These successive steps are illustrated below.

1. *The Statement.* — To bisect a given straight line.



2. Given the straight line AB.
To bisect the line.

3. *The Construction.* — From A and B as centers, with a radius greater than one-half AB, describe arcs intersecting at C and D.

Connect C and D with a straight line.

CD bisects AB.

4. *Proof.* — Two points equally distant from the extremities of a straight line determine the perpendicular bisector of that line.

How to Attack an Original Exercise.

(57) The first requisite is self-confidence. The pupil who has ordinary common sense, who has had a reasonable

amount of previous mental discipline in the grasping of abstract mathematical conceptions, who is able to command the data back of, and involved in, a given exercise, can succeed in its mastery, if he can only be made to believe in himself, and can be induced to put forth the necessary will-power, time, and effort. While it must be admitted that there are differences in minds as to their power to grasp and deal with the abstractions of geometry, mathematical ability is not so much a matter of native genius as of accumulated energy and skill acquired by persistent effort. Success here as everywhere is the result of an inflexible resolution to succeed.

Let the tyro in the subject begin at once to form the habit of honest and earnest endeavor to perform without assistance every exercise as it is presented. Time spent in such endeavor, even without success, is not wasted. The best mathematical discipline is that which is derived from the mastery of problems that tax to the utmost the energies of the mind. For disciplinary value, one hour of such study is worth a week of mechanical solutions of simple problems which require no mental effort.

No definite methods of attacking a problem, which will fit all cases, can be laid down to guide the student. These general suggestions will be useful.

If required to prove that two magnitudes are equal, try first the method of superposition.

If it is required to prove the converse of a demonstrated theorem, employ the reductio ad absurdum.

(58) Make your researches exhaustive, confining them, however, to the previous data which bear upon the problem in hand. For example, if required to prove something in regard to the trapezoid, recall everything — if necessary, read everything — in previous studies and discussions

which bears directly or indirectly on the trapezoid, endeavoring in every possible way to make the facts and principles throw light upon the theorem to be demonstrated or the problem to be solved.

(59) There are certain series of geometrical principles, which if put in synoptical form, will greatly aid the pupil in determining the limits of the range of investigation required in a given exercise. Some of the more important of these are given in articles 61, 62, and 63.

(60) After the learner has demonstrated these principles and infixed them in his memory, he has at command tests by which he may discover for himself many additional principles. Suppose, for example, he seeks to prove that The diagonal of a quadrilateral whose opposite sides are equal, divides it into two equal triangles. Applying successively the tests found in the succeeding article, he will fail with the first and the second, but will succeed with the third. By like process, all the propositions in articles 64, 65, and 66, may be readily demonstrated. Such exercise will go far toward the forming of the habit of original geometrical investigation on the part of the pupil.

(61) General Tests of the Equality of Two Triangles.

Two triangles are equal if they have respectively equal, each to each:

(1) Two sides and the included angle;
(2) Two angles and the included side;
(3) The three sides.

(62) Tests of the Equality of Two Right Triangles.

Two right triangles are equal if they have respectively equal each to each:

(1) The hypothenuse and a leg;
(2) The two legs;
(3) The hypothenuse and an acute angle;
(4) A leg and an acute angle.

63. Tests of a Parallelogram.

A quadrilateral is a parallelogram if:

(1) Its opposite sides are parallel; (Definition.)
(2) Its opposite sides are equal;
(3) A pair of opposite sides are equal and parallel;
(4) Its diagonals bisect each other;
(5) Its opposite angles are equal.

64. Exercises.

Application of tests; Article (61).

1. The three triangles formed by the meeting of the bisectors of the angles of an equilateral triangle, are equal.

2. The opposite pairs of triangles formed by the meeting of the diagonals of a parallelogram, are equal.

3. The two triangles formed by drawing lines from two opposite angles of a parallelogram to the middle points of the opposite sides, are equal.

4. The two triangles formed by drawing straight lines from the middle point of the base of an isosceles triangle to the middle points of the equal sides, are equal.

5. The triangles formed, with the base, by the bisectors of the equal angles of an isosceles triangle, are equal.

6. The triangles formed, with the base, by drawing the medians of an isosceles triangle, are equal.

7. The four triangles formed by connecting the middle points of all the sides of a given triangle, are equal.

8. *E*, *F*, *G*, and *H* are respectively the middle points of the sides *AB*, *BC*, *DC*, and *AD* of the parallelogram *ABCD*; draw *EF* and *HG*, and prove the triangles *EBF* and *HDG* equal.

9. Two isosceles triangles which have the base and the angle opposite the base of one equal respectively to the base and the angle opposite the base of the other, are equal.

10. Two isosceles triangles which have the base and a side of the one equal respectively to the base and a side of the other, are equal.

(65) Prove that the following right triangles are equal. (See Article 62.)

1. Two triangles formed by connecting the middle points of the adjacent sides of a rectangle.

2. The two right triangles formed by dropping perpendiculars from the vertices of a pair of opposite angles of a parallelogram, to the sides — or sides produced — opposite those angles.

3. The triangles formed by dropping perpendiculars from any point in the bisector of an angle, to the sides of the angle.

4. The right triangles formed by dropping perpendiculars from the extremities of the shorter of the parallel sides of an isosceles trapezoid, to the longer of the parallel sides.

5. The right triangles formed by dropping perpendiculars from the vertices of the equal angles of an isosceles triangle, to the opposite sides.

6. The four triangles formed by drawing the diagonals of a rhombus.

7. The eight triangles formed by drawing the diagonal and the diameters of a rectangle.

8. The four right triangles formed by joining the middle points of all the sides of an isosceles right triangle.

(66) Prove: (See Article 63.)

1. The quadrilateral formed by joining all the extremities of two straight lines that bisect each other, is a parallelogram.

2. The quadrilateral having each of its pairs of adjacent angles in the ratio of 1 to 2, is a parallelogram.

3. The quadrilateral formed by joining all the middle points of the adjacent sides of a quadrilateral, is a parallelogram.

4. The quadrilateral formed by joining all the middle points of the adjacent sides of a trapezium, one of the angles of which is re-entrant, is a quadrilateral.

5. D, E, and F are respectively the middle points of the sides of the triangle ABC; join DE and EF, and prove the figure $BDEF$ a parallelogram.

6. CD and AE, meeting at O, are respectively medians to the sides AB and BC of the triangle ABC; bisect AO at F, and CO at G; join DE, FG, EG, and DF, and prove that the quadrilateral $DEGF$ is a parallelogram.

7. The figure formed by producing all the bisectors of the adjacent angles of a rhomboid until they meet is a rectangle.

8. E and F are the middle points of the sides AB and DC of the parallelogram $ABCD$; join DE and BF, and prove that the quadrilateral $EBFD$ is a parallelogram.

9. The figure formed by joining the middle points of all the adjacent sides of a rhombus is a rectangle.

10. The figure formed by joining the middle points of all the adjacent sides of a rectangle is a rhombus.

(67) For exercises like the above, in determining the similarity of two triangles, see Art. 12.

CHAPTER III

Original Construction.

68. Loci.

Loci fill so important a place in the solution of geometrical problems that the meaning and use of the term should be clearly apprehended at the outset. A Locus is a line, or a series of lines, which contain all the points that possess a common property. The conception of the term is best presented to the pupil inductively. Suppose it is required to find the locus of a point equi-distant from two given points. Placing the points before the pupil, ask him to find a point equi-distant from each. This he will readily do. Ask him to find another point possessing the same property; then another, and another. After a sufficient number have been found, ask him to find all the points which possess this property. In a preliminary drill like the above, the use of the synonymous term, "place," instead of "locus," will be useful. When the pupil has discovered and fixed the line which determines the locus, he is ready for the term and its definition. To fix the meaning, he should have a great variety of consecutive and graded exercises in constructing loci, like the following.

(69) Find the locus of a point:

1. At a given distance from a given point.
2. At a given distance from a given straight line.
3. At a given distance from a given circumference.

4. Equi-distant from two given points.

5. Equi-distant from two given parallel lines.

6. Equi-distant from two given intersecting straight lines.

7. Equi-distant from two given concentric circumferences.

8. Find the locus of the centers of all circles which touch a given straight line at a given point.

9. Find the locus of the centers of all circles which touch a given circumference at a given point.

10. Find the locus of the centers of all chords of a given length which can be drawn in a given circle.

11. Given the hypothenuse of a right triangle, to find the locus of the vertices of the right angle.

12. Given the base of a triangle and its altitude, to find the locus of the vertex of the angle opposite the base.

13. Given the base of a triangle and the median drawn to the base, to find the locus of the vertex of the angle opposite the base.

14. Given the base of a triangle and the angle opposite the base, to find the locus of the vertex of the given angle.

15. Given the diagonals of a parallelogram and their point of meeting, to find the loci of its vertices.

16. Given the medians of a triangle and their point of meeting, to find the loci of its vertices.

17. Find the locus of the middle points of all chords which can be drawn through a given point within a circle.

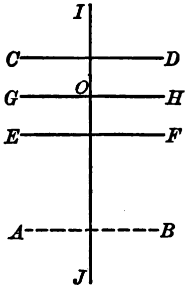
18. Find the locus of the middle points of all chords which when produced will pass through a given point without the circumference.

19. Find the locus of all points from which equal tangents can be drawn to two tangent circles.

70. Intersection of Loci.

After the pupil, by exercises like the above, has become quite familiar with the construction of loci, he should be introduced at once to the many important problems involving the intersection of loci. By illustrations with simple problems he should be brought to apprehend clearly the principle that the point of meeting of two loci having different properties possesses all the properties of both loci. For example:

Given two points and two parallel straight lines, to find a point equally distant from the points and also equally distant from the lines.



Let A and B represent the points, and CD and EF the parallel straight lines. IJ is the locus of all points equi-distant from A and B; GH is the locus of all points equi-distant from CD and EF. O, the point of meeting of GH and IJ, possesses the property of being equi-distant from A and B; also the property of being equi-distant from CD and EF.

(71) In solving problems involving the intersection of loci, the use of different colored pencils (or of different colored crayons if the blackboard is employed) with which to represent the meeting loci will be found of great service to the learner.

The solution of each of the problems below involves the intersection of two of the loci indicated in the exercises under Article 69. For example, Problem 1 involves Problems 4 and 6.

72. Exercises. Intersection of Loci.

1. To find a point equi-distant from two given points and also equi-distant from two intersecting straight lines.

Find all the points in this and the following problems which fulfill all the conditions required.

2. To find a point equi-distant from two given points and also equi-distant from two concentric circumferences.

To find a point at a given distance from a point *A*, also:

3. At a given distance from a given point.
4. At a given distance from a given straight line.
5. At a given distance from a given circumference.
6. Equi-distant from two parallel straight lines.
7. Equi-distant from two intersecting straight lines.
8. Equi-distant from two concentric circumferences.

To describe a circumference which shall touch:

9. A given line at a given point *A*, and pass through a given point *B*.

10. A given line *AB* at a given point *P*, and touch a given line *CD*.

11. Two given parallel straight lines, and pass through a given point between the lines.

12. Two parallel lines and a transversal.

13. A given circumference at a given point *A*, and pass through a given point *B*.

14. The sides of a given angle, and touch one of the sides at a given point.

15. A given circumference at a point A, and touch a given straight line.

16. With a given radius, to describe a circumference which shall touch two given circumferences.

To construct a triangle, having given:

17. The base, the altitude, and an angle at the base.

18. The base, the altitude, and a side.

19. The base, the angle opposite the base, and the altitude.

20. The base, the angle opposite the base, and the median line to the base.

21. The base, the angle opposite the base, and a side.

(73) When involved original constructions are required, it is often useful to draw the figure as though the problem were already solved, and by a careful study of the parts of the figure and their relations, to find the key to an original solution.

CHAPTER IV

Limits.

(74) Clear and accurate definitions of the terms Constant, Variable, Increasing Variable, Decreasing Variable, Limit, Superior Limit, Inferior Limit, with many and varied illustrations of their meaning and application, are essential to prepare the pupil to employ the theory of Limits intelligently in the demonstration of theorems.

(75) The idea of a variable and its limit is best presented by an exercise like the following: Upon a straight line *ab*, a variable point *v*, is moved from *a* towards *b*. During the first minute it moves half the distance from *a* to *b*, the second minute half the remaining distance, the third minute half the remaining distance, and so on. When will *v* reach *b*? Is *ab* a constant or a variable? The line *av*? The line *vb*? In what respect does the variable *av* differ from the variable *vb*? Can *v* have such a position that *av* shall equal *ab*?

(76) Exercises like the following upon the axioms of Limits will familiarize the learner with the terms involved and assist him in the mastery of the subject.

Determine whether each of the results indicated below is a constant or a variable:

1. The sum of a constant and a variable.
2. The difference of a constant and a variable.
3. The product of a constant and a variable.
4. The ratio between a constant and a variable.
5. The sum of two increasing variables.

6. The sum of two decreasing variables.
7. The sum of an increasing and a decreasing variable.
8. The difference of two increasing variables.
9. The difference of two decreasing variables.
10. The product of two increasing variables.
11. The product of two decreasing variables.
12. The product of an increasing and a decreasing variable.

(77) The most important principle in the theory of Limits is this:

If two variables are always equal, their limits are equal.

The truth of the principle is shown by superposing one variable upon the other, thus making the two, one variable, and showing that, consequently, it can have but one limit. Great pains must be taken to infix clearly and firmly in the pupil's mind this fundamental principle. He must be trained, in the process of a given demonstration involving limits, to accomplish three things:

1. He must be able to hold in mind through all the successive steps the two variables which are always equal.

2. He must conceive of them as changing, increasing, or decreasing, in accordance with the law of approach, toward the determined limit.

3. He must see and keep in mind the limits towards which the equal variables approach.

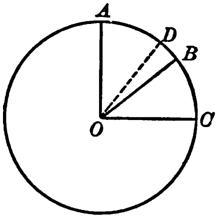
To the pupil who can do all this, the proof of a proposition by means of Limits is no more difficult than the demonstrations of theorems by other methods; but much drill and many illustrations and repetitions of principles may be necessary before this ability is acquired.

(78) The proposition proved below is the one by which learners of geometry are usually introduced to the theory of Limits. The demonstration is given in order to indicate

some methods by which the difficulties of the subject may be explained and simplified.

(79) In the same circle or in equal circles, central angles have the same ratio as their intercepted arcs.

Case 2. When the arcs are incommensurable.



The Construction. In the circle whose center is O, let the arcs AB and BC, intercepted by the central angles AOB and BOC, be incommensurable.

To prove that $\frac{\text{arc } AB}{\text{arc } BC} = \frac{<AOB}{<BOC}$.

Divide BC into a number of equal parts. One of these parts may be applied to AB a certain number of times with a remainder DB.

Since the arcs AD and BC are commensurable,

$$\frac{\text{arc } AD}{\text{arc } BC} = \frac{<AOD}{<BOC}.$$

If the unit of measure be diminished indefinitely, the remainder DB will be likewise diminished, the arc AD will approach the arc AB as its limit, and the $<AOD$ will approach the $<AOB$ as its limit. But whatever the position of D, the variable ratio $\frac{\text{arc } AD}{\text{arc } BC}$ remains equal to the variable ratio $\frac{<AOD}{<BOC}$, $\therefore \frac{\text{arc } AB}{\text{arc } BC} = \frac{<AOB}{<BOC}$.

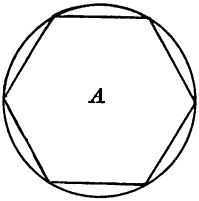
If two variables are always equal, their limits are equal. Q. E. D.

Here the two variables which are always equal are the ratios $\frac{\text{arc } AD}{\text{arc } BC}$ and $\frac{<AOD}{<BOC}$. The limits to which they approach are respectively $\frac{\text{arc } AB}{\text{arc } BC}$ and $\frac{<AOB}{<BOC}$. The law

usually employed in changing the values of the equal variables is to measure one of the arcs as BC, and to apply the measure used as far as possible to AD; then to apply a measure half as great to each of the arcs and note the corresponding change in the position of the point D, and of the line OD; then to use in like manner a measure one-fourth as large, one-eighth as large, and so on, noting the successive changes in the position of OD, and the corresponding changes in the values of the variable ratios. This exercise should be supplemented by many and varied drills like the following.

Change the lettering of the figure and require the pupils to name or write the two variables which are always equal. Require the same after changing the relative size and position of the incommensurable arcs. Require the same after constructing two equal circles and indicating an arc in each. It will often assist the learner to place a ruler upon the O and move it slowly toward B, thus allowing him to see the changes in the arc, the subtended angle, and the consequent changes in the equal ratios.

(80) Additional demonstrations by means of limits are given below, to indicate its application to surfaces and solids.



(81) To derive a formula for finding the area of a circle in terms of its radius:

Inscribe in the circle A, a regular polygon. Represent the perimeter of the inscribed polygon by p, and its apothem by a, and its surface by S.

Then, $S = \frac{1}{2} pa$. The area of a polygon is equal to the product of one half its perimeter by its apothem.

By successively doubling the number of sides of the inscribed polygon, its perimeter will approach the circumference of the circle as its limit, and the apothem will approach the radius of the circle as its limit. But, however many the number of sides, S will always equal $\frac{1}{2}\, pa$. But the limit of S is A (the area of the circle), the limit of p is $2\,\pi\, R$ (the circumference of the circle), and the limit of a is R (the radius of the circle), therefore

$$A = \tfrac{1}{2}\, R\, (2\,\pi\, R) = \pi\, \mathrm{R}^2.$$

(82) It should be carefully noted in the above that the two variables which are always equal are S and $\frac{1}{2}\, pa$ (the square contents of the inscribed polygon, and the numerical product of one-half the length of the perimeter by the apothem).

(83) To derive a formula for finding the volume of a frustum of a cone:

Inscribe in a frustum of a cone, a frustum of a regular pyramid.

Represent the lower base of the inscribed frustum by B, the upper base by b, and its altitude by A. By successively doubling the number of lateral faces, the volume of the inscribed frustum becomes an increasing variable, and the volume of the frustum of the cone is its limit. But the volume of the inscribed frustum equals

$$\tfrac{1}{2}\, a\, (B + q + \sqrt{Bb}).$$

The limit of B is $\pi\, R^2$, the limit of b is $\pi\, r^2$, therefore V (the volume of the frustum of the cone)

$$= \tfrac{1}{2}\, A\, (\pi\, R^2 + \pi\, r^2 + \sqrt{\pi\, R^2\, \pi\, r^2}).$$

Whence, $V = \frac{1}{3}\, A\, (R + r + Rr)$.

Here, as in the preceding demonstration, the two always equal variables are the two members of the equation, $V = \frac{1}{3}\, A\, (B + b + \sqrt{Bb})$.

CHAPTER V

Figures in Solid Geometry.

(84) The teacher of solid geometry often hears from the lips of pupils who have little artistic skill, expressions like this: "I can demonstrate the proposition, but I cannot draw the figure." The difficulty is a real one on the learner's part, and is best met by putting the pencil or crayon in his hand, giving him explicit directions, step by step, in the process of construction, and thus surprising him into the drawing of the figure which, without such aid, he had been unable to produce. The writer presents below some directions — covering all the more difficult figures in the geometry of solids — which he has found useful to the end proposed. They are written, not for the eye of the professional instructor in mathematical drawing, but for the aid of the teacher of geometry, with the hope that they will prove at least suggestive.

(87) To represent two parallel planes:

Draw a horizontal line AB. Draw upwards from A and B, two short, equal, straight lines AC and BD, slightly slanting toward each other from the vertical. Complete the trapezoid $CDBA$. Construct, beneath, another trapezoid $FGHE$, making the sides EH, EF, HG, and FG respectively parallel and equal, approximately, with AB, AC, BD, and CD. Shade the lines AB and EH.

(88) To represent a plane perpendicular to another plane:

Draw a trapezoid (as *CDAB* in 87). Draw *EF*, a slightly oblique line connecting the parallel bases. Draw *GE* and *FH* respectively perpendicular to *CD* and *AB* at *E* and *F*. Make *GE* and *HF* equal. Draw *GH*. Shade *AB* and *FH*.

(89) To represent two parallel planes cut by a third plane:

Construct two parallel planes (as *CDAB* and *GHEF* in 87). Draw two oblique parallel lines *IJ* and *KL*, cutting all the parallel bases of the trapezoids. Mark the intersection of *IJ* and *CD*, *M*; of *IJ* and *GH*, *O*; of *KL* and *AB*, *N*, and of *KL* and *EF*, *P*. Join *MN* and *OP*. Make *IM* and *KN* equal; also *JO* and *LP* equal. Join *IK* and *JL*. Shade *AB*, *EF*, and *KL*.

(90) To represent two intersecting planes, both perpendicular to a third plane:

In the plane *CDAB* (87), draw *FG* parallel to *AB*. Through *O*, the center of *FG*, draw within the plane an oblique line *HI*. At *O* erect a line *EO* perpendicular to *FG*. Through *E*, draw *ML* parallel and equal to *FG*. Through *E*, draw *JK* parallel and equal to *HI*. Draw *MF*, *KI*, *LG*, and *JH*. Shade lines nearest in perspective.

(91) To represent a dihedral angle:

Construct an angle *ABC*. From some point *E*, draw *ED* parallel and equal to *BA*. Draw *EF* parallel and equal to *BC*. Draw *BE*, *AD*, and *CF*.

(92) To represent a dihedral angle bisected by a plane:

Represent the dihedral angle *A-BE-F* (as in 91). Draw *BG* bisecting the angle *ABC*, also *EH* bisecting the angle *DEF*. Make *BG* equal to *EH*. Draw *GH*.

(93) To represent in a single figure an oblique prism and a right prism whose base is a right section of the oblique prism and whose edge is an edge of the oblique prism:

Draw five parallel oblique lines. From a point *A* near the lower extremity of the first line at the left, draw an oblique line *AB* from the first to the third, an oblique line *BC* from the third to the fifth, an oblique line *CD* from the fifth to the fourth, an oblique line *DE* from the fourth to the second, and join *EA*, completing the polygon *ABCDE*. Erase all parts of lines projecting below this polygon. Measure a distance *AF* on the first line; measure an equal distance *BG* on the third; an equal distance *CH* on the fifth; an equal distance *DI* on the fourth, and an equal distance *EJ* on the second. Draw *FG*, *GH*, *HI*, *IJ*, and *JF*, forming the polygon *FGHIJ*. From a point *K* on *AF*, between *A* and *F*, draw *KL* perpendicular to *BG*, *LM* nearly perpendicular to *CH*, *MN* nearly perpendicular to *DI*, *NO* perpendicular to *EJ*, and join *OK*. With *AF* as a measure, lay off *KP* on the first line, *LQ* on the third, *MR* on the fifth, *NS* on the fourth, and *OT* on the second. Join *PQ*, *GR*, *RS*, *ST*, and *TP*, forming the polygon *PQRST*. Erase all lines projecting above this polygon. Shade lines nearest in perspective.

(94) To construct between two parallel planes a right, an oblique, and a rectangular parallelopiped:

Construct the right parallelopiped *ABCDEFGH*, making *AB* and *CD* the horizontal edges of the upper base, and *EF* and *GH* the corresponding edges of the lower base. Produce *AB*, *CD*, *EF*, and *GH* to the right. In the plane *CH* produced, draw an oblique line *KO*. With *CA* as a radius from *K*, strike an arc cutting *AB* produced in *I*. Draw *IM* parallel to *KO*. Join *IK* and *MO*. With *AB* as a measure, lay off on the edges produced *IJ*, *KL*, *MN*, and *OP*. Join *JN*, *NP*, *PL*, and *LJ*. Erase all lines projecting to the right of the figure *JNPL*. Produce *AC*, *BD*, *EG*, and *FH*. Erect perpendiculars *HQ* and *GR* from *H* and *G* respectively to *AC* produced and *BD* produced. Join *QR*. Using *AC* as a measure, lay off on the produced edges from *Q*, *R*, *H*, and *G* respectively *QS*, *RT*, *HV*, and *GU*. Join *ST*, *VS*, *VU*, and *UT*. Erase all lines projecting beyond the figure *STVU*.

(95) To represent a series of triangular prisms inscribed in a triangular pyramid:

Divide *AB*, an edge of the triangular pyramid *A-BCD*, into any number of equal parts. From *G*, the point of division nearest the base, draw *GI* parallel to *BD* to meet *AD* at *I*; draw *GH* parallel to *BC* to meet *AC* at *H*. From *I*, draw *IJ* parallel to *GB* to meet *BD* at *J*, and from *H*, draw *HK* parallel to *GB* to meet *BC* at *K*. Draw *HI* and *JK*. In like manner form successively the remaining prisms.

(96) To represent a regular dodecahedron:

Construct a regular pentagon, *ABCDE*. From *A*, in a course directly from the center of the pentagon, draw *AF*, making *AF* a little shorter than the line *AB*. In like manner draw *BH* equal to *AF*. Complete the regular pentagon *FABHG*. In like manner construct a regular pentagon upon each side of *ABCDE*. To make the figure appear convex in perspective, shade the original pentagon. To produce a concave effect, shade the outer edges of the figure.

(97) To represent a regular icosahedron:

Construct an equilateral triangle *ABC*. Bisect *AB* at

D, *AC* at *E*, and *BC* at *F*. From *D* draw *DG* perpendicular to *AB* and a little shorter than *AD*. In like manner draw *EH* and *FI* from *AC* and *BC* respectively. Draw *DE*, *EF*, *FD*, *AH*, *HC*, *CI*, *IB*, *BG*, and *GA*. Shade the triangle *DEF*.

(98) To represent a sphere in outline:

Construct a circle. Draw a vertical diameter *AB* and a horizontal diameter *CD*. Connect *A* and *B* by an arc *AOB*, drawn to the right and within the original circle. Draw a corresponding arc *AQB* to the left. In like manner connect *C* and *D* by an arc *CMD* below and an arc *CND* above the horizontal diameter. Shade *AOB* and *CMD*.

(99) To represent a solid sphere:

Draw a circle. Fix a point in the circumference as a pole. Conceive a meridian to be drawn through the pole at an angle of about sixty degrees. To the right of this meridian shade heavily and evenly to the semi-circumference. To the left make the shade lighter and lighter until it finally disappears, with the exception of a light shading about the semi-circumference at the left.

(100) To represent a sphere inscribed in a cube:

Draw a cube. Fix the middle points of the upper and lower bases and of the lateral sides at the right and left. Pass a circle through these points. Represent the sphere as in 98 or 99.

By a slight variation from this device a sphere may be represented as inscribed in a cylinder, a pyramid, a cone, and other of the more regular forms of geometrical solids.

ANSWERS

Art. 1.

1. $x = 2$ or -1
2. $x = 2$ or -9
3. $x = 3$ or $-5\frac{1}{2}$
4. $x = 1$ or $-4\frac{2}{3}$
5. $x = -1 \pm 2\sqrt{2}$
6. $x = -\dfrac{5 \pm \sqrt{69}}{2}$
7. $x = -\dfrac{9 \pm \sqrt{185}}{4}$
8. $x = \dfrac{2 \pm \sqrt{7}}{3}$
9. $x = a$ or $-6a$
10. $x = 1$
11. $x = 25$ or 36
12. $x = 3$ or 8
13. $x = 54$ or $\frac{2}{3}$
14. $x = -\dfrac{3b \pm b\sqrt{5}}{2}$
15. $x = \pm 5$ or $\pm \sqrt{133}$

Simultaneous Equations.

1. $x = 2$ or 3; $y = 3$ or 2
2. $x = 6$ or -1; $y = 1$ or -6
3. $x = 4$ or 5; $y = 5$ or 4
4. $x = 4$ or -1; $y = 1$ or -4
5. $x = 2$ or 1; $y = 1$ or 2
6. $x = \frac{1}{2}\,(b \pm \sqrt{b^2 - 4a}$; $y = \frac{1}{2}\,(b \mp \sqrt{b^2 - 4a}$
7. $x = -\frac{1}{2}\,(b \pm \sqrt{b^2 - 4a}$; $y = -\frac{1}{2}\,(b \mp \sqrt{b^2 - 4a}$
8. $x = m$ or 0; $y = 0$ or m
9. $x = P$ or $2P$; $y = 2P$ or P
10. $x = \dfrac{a^2 + b}{2a}$; $y = \dfrac{a^2 - b}{2a}$

Art. 3.

3. 170°; 63°
4. 60°; $67\frac{1}{2}$°
5. 49°, 41°
6. 103°, 77°
7. $90^\circ + \dfrac{a^\circ}{2}$, $90^\circ - \dfrac{a^\circ}{2}$

Art. 4.

1. 98°
2. 96°, 84°
3. 36°, 72°, 108°, 144°
4. $51\frac{3}{7}$°, $77\frac{1}{7}$°, $102\frac{6}{7}$°, $128\frac{4}{7}$°

5. 72°, 72°, 72°, 144°
6. $87\frac{33}{41}$°, $92\frac{8}{41}$°, $136\frac{4}{41}$°, $43\frac{37}{41}$°
7. 144°, 72°, 48°, 96°
8. 24°, 48°, 96°, 192°
9. 80°, 40°, 60°, 180°
10. 66°, 116°, 56°, 122°
11. 30°, 120°, 150°, 60°
12. 55°, 35°, 105°, 165°

Art. 5.

1. 15°, 45°, 120°
2. 50°, 60°, 70°
3. $56\frac{2}{3}$°, 60°, $63\frac{1}{3}$°
4. $37\frac{11}{37}$°, $50\frac{10}{37}$°, $92\frac{16}{37}$°
5. $83\frac{1}{13}$°, $55\frac{5}{13}$°, $41\frac{7}{13}$°
6. 80°, 70°, 30°
7. 40°, 80°, 60°
8. 18°, 90°, 72°
9. 10°, 70°, 100°[1]
10. 24°, 12°, 144°
11. 79°
12. 122°
13. 140°
14. 34°, 54°
15. 135°
16. 22°
17. 34°
18. 45°

Art. 6.

1. 84°
2. 45°
3. 60°
4. $(90-m)$°
5. 31°
6. $118\frac{1}{2}$°
7. 90°
8. 94°; 99°; 140°; 175°
9. 23°, 44°
10. 84°
11. $127\frac{1}{2}$°, $127\frac{1}{2}$°, 105°
12. 160°, 10°, 10°
13. 4°, 70°, 70°
14. 120°

Art. 7.

2. 30
3. 14
4. 93°
5. 68°
6. $128\frac{4}{7}$°
7. { 4 of the <'s 12° each
   4 of the <'s 78° each
8. { *EDF*, 45°; *EFD*, 45°; *ABE*, $67\frac{1}{2}$°
   AEB, $67\frac{1}{2}$°; *EBF*, $22\frac{1}{2}$°
   BEF, $22\frac{1}{2}$°; *EFB*, 135°
10. 36°, 108°, 108°, 144°, 140°
11. 60°, 72°, 84°, 144°
12. 152°, 104°, 104°
13. 8°, 172°, 172°

Art. 8.

1. $\frac{n(n-1)}{2}$
2. Formula, $\frac{n(n-3)}{2}$

Art. 9.

3. 58°, 90°, 212°
4. 154°, 108°, 98°
5. 50°
6. 17°
7. 12°
8. 69°
9. 56°, 76°, 48°
10. 69°, 55°, 56°
11. 100°, 30°, 50°
12. 63°, $60\frac{1}{2}$°, $56\frac{1}{2}$°
14. 22
15. 15, 13, 14
16. { $27\frac{1}{2}$, $22\frac{1}{2}$, $17\frac{1}{2}$
    39, 37, 33
18. 141°, 81°

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[1] In answers to concrete problems, only positive results are given.

Art. 14.

1. 18
2. $3\frac{6}{17}$
3. 60
4. $4\frac{4}{5}$
5. 70
6. 15
7. $18\frac{3}{4}$
8. 24
9. $9\frac{3}{5}$
10. 28
11. $21\frac{3}{5}$
12. 12

Art. 15.

1. 12
2. $15\sqrt{15}$
3. $mn\sqrt{mn}$
4. $3\sqrt{3}$
5. 16
6. 6
7. $8\frac{1}{8}$
8. 18
9. 2, 18
10. 28

Art. 17.

4. $4(\sqrt{5}-1)$
5. $5(\sqrt{5}-1)$
6. $\frac{a}{2}(\sqrt{5}-1)$

Art. 18.

1. 1012
2. $39\frac{1}{16}$
3. 8
4. 3
8. 147
9. 8
10. $9m$

Art. 19.

1. 378
2. 3
3. 84, 54
5. $18\frac{2}{3}$
6. 84
7. $\frac{bc}{a-b}$
8. 216
9. 8
10. $\frac{m}{n}$

Art. 23.

7. 2
8. $8x$
9. 57
10. 21, 28
11. 15, 9
12. 30, 24
13. 12, 16, 20
14. 24, 32, 40
15. 40
16. (1) 6, (2) 48
17. (1) $4\sqrt{5}$, (2) $16\sqrt{5}$
18. $8\sqrt{2}$
19. $a\sqrt{2}$
20. (1) $3\sqrt{3}$, (2) $9\sqrt{3}$
21. (Last) $\frac{a^2}{4}\sqrt{3}$
22. $12\sqrt{3}$
23. $\frac{2a}{3}\sqrt{3}$
24. $\frac{5}{2}\sqrt{7}$
25. $300\sqrt{3}$
26. 18, 108
27. 240
28. $17\sqrt{35}$
29. 1280
30. 288
31. $\frac{d^2}{2}$
32. $72\sqrt{3}$
33. $20\sqrt{2}$

Art. 28.

1. (Last) $40000\sqrt{5}$
2. $6 + 6\sqrt{39}$
3. $24 + 20\sqrt{3}$
4. $12\sqrt{22} + 60\sqrt{7} + 24\sqrt{35}$
5. $9\frac{3}{5}$
6. $10\frac{10}{11}\sqrt{2}$
9. $2\sqrt{21}$

Art. 29.

4. $24\sqrt{3}$
5. $\frac{2}{3}\sqrt{3}$
6. $16\sqrt{3}$
7. $128 + 128\sqrt{2}$
8. $576 + 576\sqrt{2}$
9. $4n^2 + 4n^2\sqrt{2}$

Art. 30.

9. 3 : 1
10. 4 : 3
12. $2 : \sqrt{3}$
13. 4 : 3

Art. 31.

1. $75\sqrt{3}$
2. 200
3. $150\sqrt{3}$
4. $200\sqrt{2}$
5. 300
6. 36π
7. $\frac{100}{\pi}$
8. $40\pi\sqrt{5}$
9. $33\frac{1}{3}\pi$
10. 16π
11. $25\sqrt{3} - 50\pi$
12. $36\sqrt{3} - 72\pi$

Art. 35.

1. 782
2. 864
3. $\frac{4}{5}$
4. $480\sqrt{3}$
5. $72\sqrt{3}$
6. 5, 6, 7
7. 2, 4, 6
8. 44
9. 288

Art. 36.

1. $2\sqrt{2}$
2. $12\sqrt[3]{3\pi}$
3. 276
4. $54\sqrt{2}$
5. $72\sqrt{3}$
6. $45\sqrt{3}$
7. $168\sqrt{6}$
8. $48\sqrt{21}$
9. $2400\sqrt{3}$
10. $\frac{3a^2b}{2}\sqrt{3}$
11. $\sqrt{3} : 4$
12. $4\sqrt{3}$

Art. 37.

1. 630
2. $24\sqrt{21}$
3. $324\sqrt{3}$
4. 8
5. $64\sqrt{15}$
6. 336
7. 349, 44
8. 18
9. $16\sqrt{65}$
10. $20\sqrt{41}$

Art. 38.

1. 140
2. $\frac{33}{2}\sqrt{39}$
3. $\frac{128}{3}\sqrt{3}$
4. $512\sqrt{2}$
5. $3888\sqrt{3}$
6. $\frac{50}{3}\sqrt{167}$

Art. 39.

1. 1760
2. 7168
3. $624\sqrt{35}$
4. $7\frac{1}{3}$
5. 30

Art. 40.

1. 1351
2. 50628
3. 3774
4. 4
5. 896
6. $2\sqrt[3]{21}$, $6-2\sqrt[3]{21}$
7. $2602\frac{2}{3}\sqrt{5}$

Art. 41.

1. 96; $\frac{288}{\pi}$
2. 192π; 768π
3. 8192π
4. 31433π
5. $\frac{36864}{\pi}$ ft.
6. 42π
7. 196π
8. 784π
9. 62500π

Art. 42.

1. 200π; $333\frac{1}{3}\pi\sqrt{3}$
2. 500π, 2000π
3. $32\sqrt{2}\pi$
4. $333\frac{1}{3}\pi\sqrt{3}$

Art. 43.

1. 240π
2. $\frac{60}{\pi}$
3. 10

Art. 44.

1. 784π; $3658\frac{2}{3}\pi$
2. $\frac{5}{\pi}\sqrt{10\pi}$
3. 40000
4. 3
5. $166\frac{2}{3}\pi$
6. $8\sqrt{3}$
7. $5\sqrt[3]{12}$
7. $5\sqrt{12}$
8. $1:1$; $3:2$; $3:2$
9. $6:\pi$; $6:\pi$
10. $325\frac{1}{3}\pi$

Art. 45.

1. $400\sqrt{3}$
2. $384\sqrt{3}$
3. $1152 + 1152\sqrt{2}$
4. $32\sqrt{3}$
5. $4\sqrt{41}$
6. 96
7. $43\frac{23}{27}$
8. 6, 6, 18. 18, 22, 22
9. $10\sqrt[3]{3}$
10. $10\sqrt{2}$

11. $8\sqrt{31}$
12. 1536
13. $2\frac{2}{3}\sqrt{6}$
14. 100
15. $1\frac{1}{6}\pi$
16. 6, 8
17. 10
18. $45\frac{59}{60}$
19. 1188π
20. $\dfrac{12400}{49\pi}$
21. 1 : 4
22. $3\frac{3}{4}\sqrt{7}$
23. 189
24. $35\sqrt{3}$ ft.
25. $617\frac{17}{18}$
26. $10 - 5\sqrt{2}$
27. 50, 40
28. $864\pi\sqrt{3}$
29. $9 - 4\frac{1}{2}\sqrt{2}$ in., $4\frac{1}{2}\sqrt{2}$ in.
30. 125π
31. $37\frac{1}{2}\pi$ sq. in.
32. (*a*) 64000000π sq. mi.
    (*b*) $170,666,666,666\frac{2}{3}$ cu. mi.
33. 384π
34. $7\frac{91}{99}\pi$ gal.
35. $5,366,666,666,666\frac{2}{3}\pi$ cu. mi.
36. $844\pi\sqrt{5}$
39. $48\pi - 36\sqrt{3}$
40. $16\sqrt{17}$
41. $5\dfrac{5}{\pi}$
42. $2856\sqrt{6}$

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